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1-13	/65	15	/20
14	/15	16	/10
		Total	/110

MATH 221 Final Exam 505 Fall 2021

Sections 504/505 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the equation of the plane tangent to the graph of $z = f(x,y) = x^2y + xy^2$ at $(2,1)$.
Its z-intercept is

a. -6

b. -10

c. -12 Correct Choice

d. -16

e. -24

f. 6

g. 10

h. 12

i. 16

j. 24

Solution: $f(x,y) = x^2y + xy^2$ $\quad f_x(x,y) = 2xy + y^2$ $\quad f_y(x,y) = x^2 + 2xy$

$f(2,1) = 6$ $\quad f_x(2,1) = 5$ $\quad f_y(2,1) = 8$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 6 + 5(x-2) + 8(y-1) = 5x + 8y - 12 \qquad c = -12$

2. Find the equation of the line perpendicular to the hyperboloid $xyz = 6$ at the point $(1,2,3)$.

a. $x + 2y + 3z = 14$

b. $x + 2y + 3z = 18$

c. $(x,y,z) = (1+6t, 2-3t, 3+2t)$

d. $(x,y,z) = (6+t, 3-2t, 2+3t)$

e. $(x,y,z) = (1+2t, 2-3t, 3+6t)$

f. $6x + 3y + 2z = 18$

g. $6x + 3y + 2z = 49$

h. $(x,y,z) = (1+6t, 2+3t, 3+2t)$ Correct Choice

i. $(x,y,z) = (6+t, 3+2t, 2+3t)$

j. $(x,y,z) = (1+2t, 2+3t, 3+6t)$

Solution: $P = (1,2,3)$ $\quad F = xyz$ $\quad \vec{\nabla}F = (yz, xz, xy)$ $\quad \vec{N} = \vec{\nabla}F\Big|_{(1,2,3)} = (6,3,2)$

$X = P + t\vec{N} = (1,2,3) + t(6,3,2) = (1+6t, 2+3t, 3+2t)$

3. Find the area of the triangle with vertices $(1,1,1)$, $(2,2,4)$ and $(1,3,9)$.

a. 1 f. $\sqrt{2}$
b. 2 g. $2\sqrt{2}$
c. 3 h. $3\sqrt{2}$ Correct Choice
d. 4 i. $4\sqrt{2}$
e. 6 j. $6\sqrt{2}$

Solution: $\vec{u} = (2,2,4) - (1,1,1) = (1,1,3)$ $\vec{v} = (1,3,9) - (1,1,1) = (0,2,8)$

$$\vec{u} \times \vec{v} = (1,1,3) \times (0,2,8) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 3 \\ 0 & 2 & 8 \end{vmatrix} = \hat{\imath}(8-6) - \hat{\jmath}(8) + \hat{k}(2) = (2,-8,2)$$

$$A = \frac{1}{2}|\vec{u} \times \vec{v}| = \frac{1}{2}\sqrt{4+64+4} = \frac{1}{2}\sqrt{72} = 3\sqrt{2}$$

4. Find the point where the line $(x,y,z) = (3,2,1) + t(1,2,3)$ intersects the plane $x - y + z = -2$. At this point, $x + y + z =$

a. -6 Correct Choice f. 6
b. -4 g. 4
c. -2 h. 2
d. -1 i. 1
e. 0 j. none of these

Solution: $-2 = x - y + z = (3+t) - (2+2t) + (1+3t) = 2t + 2$ $t = -2$
$(x,y,z) = (3,2,1) - 2(1,2,3) = (1,-2,-5)$ $1 - 2 - 5 = -6$

5. A particle moves along the curve $\vec{r}(t) = (t,t^2,t^3)$ from $(1,1,1)$ to $(2,4,8)$ due to the force $\vec{F} = (z,y,x)$. Find the work done by the force.

a. 14 f. $\frac{45}{2}$ Correct Choice
b. 24 g. $\frac{70}{3}$
c. 36 h. $\frac{93}{5}$
d. 45 i. $\frac{96}{5}$
e. 48 j. $\frac{186}{5}$

Solution: $\vec{v} = (1,2t,3t^2)$ $\vec{F}(\vec{r}(t)) = (t^3,t^2,t)$ $\vec{F} \cdot \vec{v} = t^3 + 2t^3 + 3t^3 = 6t^3$

$$W = \int_{(1,1,1)}^{(2,4,8)} \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F} \cdot \vec{v}\, dt = \int_1^2 6t^3\, dt = \left[\frac{3t^4}{2}\right]_1^2 = \frac{48}{2} - \frac{3}{2} = \frac{45}{2}$$

6. A circuit has two resistors $R_1 = 200\ \Omega$ and $R_2 = 300\ \Omega$ in parallel.
The net resistance R satisfies $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is increasing at $2\ \Omega$/sec and R_2 is decreasing at $9\ \Omega$/sec at what rate is R changing?

a. $-\frac{9}{50}\ \Omega$/sec
b. $-\frac{18}{25}\ \Omega$/sec   Correct Choice
c. $-\frac{9}{25}\ \Omega$/sec
d. $-\frac{36}{25}\ \Omega$/sec
e. $-500\ \Omega$/sec
f. $\frac{9}{50}\ \Omega$/sec
g. $\frac{18}{25}\ \Omega$/sec
h. $\frac{9}{25}\ \Omega$/sec
i. $\frac{36}{25}\ \Omega$/sec
j. $500\ \Omega$/sec

Solution: $\frac{1}{R} = \frac{1}{200} + \frac{1}{300} = \frac{300+200}{200 \cdot 300} = \frac{1}{120}$ $R = 120$

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} \qquad \frac{dR}{dt} = \frac{R^2}{R_1^2}\frac{dR_1}{dt} + \frac{R^2}{R_2^2}\frac{dR_2}{dt} = \frac{120^2}{200^2}2 - \frac{120^2}{300^2}9 = -\frac{18}{25}$$

7. Compute $\iint x^2\,dA$ over the region between the parabolas $y = 4 - x^2$ and $y = 8 - 2x^2$.

a. $\frac{1}{15}$
b. $\frac{2}{15}$
c. $\frac{4}{15}$
d. $\frac{8}{15}$
e. $\frac{16}{15}$
f. $\frac{32}{15}$
g. $\frac{64}{15}$
h. $\frac{128}{15}$ Correct Choice
i. $\frac{256}{15}$
j. $\frac{512}{15}$

Solution: Find where the parabolas intersect: $4 - x^2 = 8 - 2x^2 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

$$\iint x^2\,dA = \int_{-2}^{2}\int_{4-x^2}^{8-2x^2} x^2\,dy\,dx = \int_{-2}^{2}\left[x^2 y\right]_{y=4-x^2}^{8-2x^2} dx = \int_{-2}^{2} x^2[(8-2x^2)-(4-x^2)]\,dx = \int_{-2}^{2} x^2(4-x^2)\,dx$$

$$= \int_{-2}^{2}(4x^2 - x^4)\,dx = \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = 2\left[\frac{32}{3} - \frac{32}{5}\right] = 64\frac{5-3}{15} = \frac{128}{15}.$$

8. Compute $\iint\limits_R \frac{1}{x^2+y^2}\,dx\,dy$ over the ring $9 \le x^2+y^2 \le 16$.

a. $2\pi\ln\frac{16}{9}$

b. $4\pi\ln\frac{16}{9}$

c. $\pi\ln\frac{4}{3}$

d. $2\pi\ln\frac{4}{3}$ Correct Choice

e. $4\pi\ln\frac{4}{3}$

f. $2\pi\ln\frac{9}{16}$

g. $4\pi\ln\frac{9}{16}$

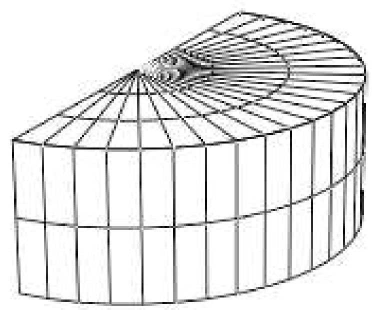
h. $\pi\ln\frac{3}{4}$

i. $2\pi\ln\frac{3}{4}$

j. $4\pi\ln\frac{3}{4}$

Solution: $\iint\limits_R \frac{1}{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_3^4 \frac{1}{r^2}\,r\,dr\,d\theta = 2\pi\ln r\Big|_3^4 = 2\pi\ln\frac{4}{3}$

9. Compute $\iiint y\,dV$ over the half cylinder $x^2+y^2 \le 9$ with $y \ge 0$ for $-1 \le z \le 1$.



a. $\frac{3\pi}{2}$

b. $\frac{9\pi}{2}$

c. 9π

d. 18π

e. 36π

f. $\frac{3}{2}$

g. $\frac{9}{2}$

h. 9

i. 18

j. 36 Correct Choice

Solution: We use cylindrical coordinates. $y = r\sin\theta$ $dV = r\,dr\,d\theta\,dz$

$$\iiint y\,dV = \int_{-1}^{1}\int_0^{\pi}\int_0^{3} r\sin\theta\,r\,dr\,d\theta\,dz = \int_{-1}^{1} dz\int_0^{\pi}\sin\theta\,d\theta\int_0^{3} r^2\,dr$$

$$= \Big[z\Big]_{-1}^{1}\Big[-\cos\theta\Big]_0^{\pi}\left[\frac{r^3}{3}\right]_0^3 = [2][2][9] = 36$$

10. Compute $\int_{(1,1,1)}^{(1/e,1/e,e^2)} \vec{F} \cdot d\vec{s}$ where $\vec{F} = (z, z, x+y)$ along the curve $\vec{r}(t) = (e^{-t}, e^{-t}, e^{2t})$.
HINT: Find a scalar potential.

a. $4 - 4e$
b. $4 - 2e$
c. $2 - 2e$
d. $2 - 4e$
e. $4 + 4e$
f. $4e - 4$
g. $2e - 4$
h. $2e - 2$ Correct Choice
i. $4e - 2$
j. $2 + 2e$

Solution: $\vec{F} = (z, z, x+y) = \vec{\nabla} f$ for $f = xz + yz$. So, by the FTCC:

$$\int_{(1,1,1)}^{(1/e,1/e,e^2)} \vec{F} \cdot d\vec{s} = \int_{(1,1,1)}^{(1/e,1/e,e^2)} \vec{\nabla} f \cdot d\vec{s} = f\left(\frac{1}{e}, \frac{1}{e}, e^2\right) - f(1,1,1)$$
$$= \left(\frac{1}{e}e^2 + \frac{1}{e}e^2\right) - (1 \cdot 1 + 1 \cdot 1) = 2e - 2$$

11. Compute $\oint (\sin(x^3) - 2x^2 y)\,dx + (2xy^2 + \cos(y^3))\,dy$ counterclockwise around the circle $x^2 + y^2 = 9$.
HINT: Use a theorem.

a. 3π
b. 6π
c. 9π
d. 12π
e. 18π
f. 27π
g. 36π
h. 54π
i. 72π
j. 81π Correct Choice

Solution: By Green's Theorem,

$$\oint (\sin(x^3) - 2x^2 y)\,dx + (2xy^2 + \cos(y^3))\,dy = \iint \left(\frac{\partial}{\partial x}(2xy^2 + \cos(y^3)) - \frac{\partial}{\partial y}(\sin(x^3) - 2x^2 y)\right) dx\,dy$$
$$= \iint (2y^2 + 2x^2)\,dx\,dy = \int_0^{2\pi}\int_0^3 2r^2 r\,dr\,d\theta = 2\pi\left[\frac{r^4}{2}\right]_{r=0}^{3} = 81\pi$$

12. Compute $\iint_{\partial C} \vec{F} \cdot d\vec{S}$ for $\vec{F} = (x^3 z, y^3 z, x^2 + y^2)$ over the total surface of the cylinder $x^2 + y^2 \le 4$ for $0 \le z \le 3$.

HINT: Use Gauss' Theorem.

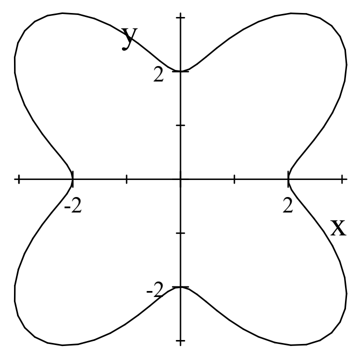
a. 6π
b. 9π
c. 12π
d. 18π
e. 27π
f. 36π
g. 54π
h. 108π Correct Choice
i. 216π
j. 432π

Solution: By Gauss' Theorem, $\iint_{\partial C} \vec{F} \cdot d\vec{S} = \iiint_C \vec{\nabla} \cdot \vec{F}\, dV$. Use cylindrical coordinates.

$\vec{\nabla} \cdot \vec{F} = 3x^2 z + 3y^2 z = 3r^2 z$

$$\iiint_C \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^2 \int_0^3 3r^2 z\, r\, dz\, dr\, d\theta = 6\pi \left[\frac{r^4}{4} \right]_0^2 \left[\frac{z^2}{2} \right]_0^3 = 6\pi(4)\left(\frac{9}{2}\right) = 108\pi$$

13. Compute $\oint \vec{\nabla} f \cdot d\vec{s}$ counterclockwise once around the polar curve $r = 3 - \cos(4\theta)$ for the function $f(x,y) = x^2 + y$.



HINT: Use the FTCC or Green's Theorem.

a. π
b. 2π
c. 4π
d. 6π
e. 0 Correct Choice
f. 8π
g. 12π
h. 16π
i. 32π
j. none of these

Solution: By the FTCC, $\int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A)$.

However, since it is a closed curve, $B = A$, and $\int_A^B \vec{\nabla} f \cdot d\vec{s} = 0$.

OR by Green's Theorem, $\oint \vec{\nabla} f \cdot d\vec{s} = \iint \vec{\nabla} \times \vec{\nabla} f \cdot \hat{k}\, dA = 0$ because $\vec{\nabla} \times \vec{\nabla} f = 0$ for any f.

Work Out: (Points indicated. Part credit possible. Show all work.)

14. (15 points) We want to make a cylindrical aluminum can with lids to hold 32π cm^3.

However, the top and bottom are to be twice as thick as the sides.

The metal for the sides is the surface area, $2\pi rh$.

The metal for each of the 2 ends is twice their area, πr^2.

So the total amount of metal is: $M = 2\pi rh + 4\pi r^2$.

Find the radius and height of the can and the amount of metal needed to make a can which uses the least amount of metal.

Solution: We minimize $M = 2\pi rh + 4\pi r^2$

subject to the constraint that the volume is $V = \pi r^2 h = 32\pi$.

The gradients are $\vec{\nabla} M = \langle 2\pi h + 8\pi r, 2\pi r \rangle$ and $\vec{\nabla} V = \langle 2\pi rh, \pi r^2 \rangle$.

The Lagrange equations are $2\pi h + 8\pi r = \lambda 2\pi rh \qquad 2\pi r = \lambda \pi r^2$

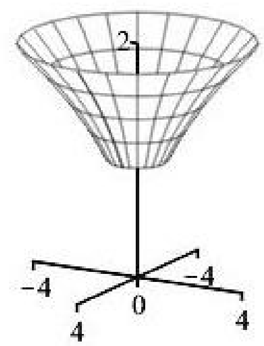
We solve for λ and equate: $\lambda = \dfrac{2\pi h + 8\pi r}{2\pi rh} = \dfrac{2\pi r}{\pi r^2}$ or $\dfrac{h + 4r}{rh} = \dfrac{2}{r}$

We multiply both sides by rh and solve: $h + 4r = 2h \qquad h = 4r$

We substitute this into the constraint: $V = \pi r^2 h = 4\pi r^3 = 32\pi \qquad r^3 = 8$

Therefore: $r = 2 \qquad h = 8 \qquad M = 2\pi rh + 4\pi r^2 = 32\pi + 16\pi = 48\pi$

15. (20 points) Verify Stokes' Theorem $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$



for the vector field $\vec{F} = \langle -yz, xz, z^2 \rangle$ and the funnel $r = z^2$ for $1 \le z \le 2$ oriented down and out.

Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the curl $\vec{\nabla} \times \vec{F}$ in rectangular coordinates.

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(-x) - \hat{\jmath}(--y) + \hat{k}(z - -z) = \langle -x, -y, 2z \rangle$

b. The funnel surface, S, may be parametrized by $\vec{R}(z,\theta) = (z^2 \cos\theta, z^2 \sin\theta, z)$.

What is $\vec{\nabla} \times \vec{F}$ on the funnel?

Solution: $\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(z,\theta)} = \langle -z^2 \cos\theta, -z^2 \sin\theta, 2z \rangle$

c. Find the normal to the funnel.

Solution: $\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_z = & (2z\cos\theta, & 2z\sin\theta, & 1) \\ \vec{e}_\theta = & (-z^2\sin\theta, & z^2\cos\theta, & 0) \end{vmatrix}$

$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(-z^2\cos\theta) - \hat{\jmath}(--z^2\sin\theta) + \hat{k}(2z^3\sin^2\theta - -2z^3\cos^2\theta) = \langle -z^2\cos\theta, -z^2\sin\theta, 2z^3 \rangle$

This is up and in. We need down and out.

Reverse: $\vec{N} = \langle z^2\cos\theta, z^2\sin\theta, -2z^3 \rangle$

d. Compute the integral $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$.

Solution: $\vec{\nabla} \times \vec{F} \cdot \vec{N} = -z^4\cos^2\theta - z^4\sin^2\theta - 4z^4 = -z^4 - 4z^4 = -5z^4$

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_1^2 \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dz\, d\theta = \int_0^{2\pi}\int_1^2 -5z^4\, dz\, d\theta = -2\pi[z^5]_1^2 = -2\pi(32-1) = -62\pi$$

Second the Right Hand Side:

e. The circle, T, at the top of the funnel may be parametrized by $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 2)$.

What is $\vec{F} = \langle -yz, xz, z^2 \rangle$ on the top circle?

Solution: $\vec{F} = \langle -8\sin\theta, 8\cos\theta, 4 \rangle$

f. What is the tangent vector to the circle?

Solution: $\vec{v} = \langle -4\sin\theta, 4\cos\theta, 0 \rangle$

In the 1^{st} quadrant, $v_1 \le 0$ and $v_2 \ge 0$. So $\vec{v}$ points counterclockwise.

By the RHR, we need $\vec{v}$ clockwise. Reverse it: $\vec{v} = \langle 4\sin\theta, -4\cos\theta, 0 \rangle$

g. Compute the integral $\oint_T \vec{F} \cdot d\vec{s}$.

Solution: $\vec{F} \cdot \vec{v} = -32\sin^2\theta - 32\cos^2\theta + 0 = -32$

$$\int_T \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -32\, d\theta = -64\pi$$

h. The circle, B, at the bottom of the funnel may be parametrized by $\vec{r}(\theta) = (\cos\theta, \sin\theta, 1)$.

What is $\vec{F} = \langle -yz, xz, z^2 \rangle$ on the bottom circle?

Solution: $\vec{F} = \langle -\sin\theta, \cos\theta, 1 \rangle$

i. What is the tangent vector to the circle?

Solution: $\vec{v} = \langle -\sin\theta, \cos\theta, 0 \rangle$

In the 1^{st} quadrant, $v_1 \le 0$ and $v_2 \ge 0$. So $\vec{v}$ points counterclockwise.
By the RHR, we need $\vec{v}$ counterclockwise. So $\vec{v}$ is good.

j. Compute the integral $\oint_B \vec{F} \cdot d\vec{s}$.

Solution: $\vec{F} \cdot \vec{v} = \sin^2\theta + \cos^2\theta + 0 = 1$

$$\int_B \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi$$

k. Compute the **TOTAL** right hand side:

Solution: $\int_{\partial S} \vec{F} \cdot d\vec{S} = \int_T \vec{F} \cdot d\vec{S} + \int_B \vec{F} \cdot d\vec{S} = -64\pi + 2\pi = -62\pi$

which agrees with (c).

16. (10 points) Find the volume and z-component of the centroid of the solid between the surfaces

$$z = (x^2 + y^2)^{3/2} \quad \text{and} \quad z = 8.$$

Solution: In cylindrical coordinates, the bottom surface is $z = (r^2)^{3/2} = r^3$.

So $r^3 \le z \le 8$ and $0 \le r \le 2$.

$$V = \int_0^{2\pi}\int_0^2\int_{r^3}^8 r\,dz\,dr\,d\theta = 2\pi\int_0^2\Big[rz\Big]_{z=r^3}^8 dr = 2\pi\int_0^2 (8r - r^4)\,dr$$

$$= 2\pi\left[4r^2 - \frac{r^5}{5}\right]_0^2 = 2\pi\left(16 - \frac{32}{5}\right) = 32\pi\left(1 - \frac{2}{5}\right) = \frac{96\pi}{5}$$

$$V_z = \int_0^{2\pi}\int_0^2\int_{r^3}^8 zr\,dz\,dr\,d\theta = 2\pi\int_0^2\left[r\frac{z^2}{2}\right]_{z=r^3}^8 dr = \pi\int_0^2 (64r - r^7)\,dr$$

$$= \pi\left[32r^2 - \frac{r^8}{8}\right]_0^2 = \pi(128 - 32) = 96\pi$$

$$\bar{z} = \frac{V_z}{V} = 96\pi\frac{5}{96\pi} = 5$$