Name________________________

MATH 221 Exam 1 Spring 2023

Section 501 Solutions P. Yasskin

1-9	/54	11	/25
10	/15	12	/10
		Total	/104

Multiple Choice: (6 points each. No part credit.)

1. Consider the sphere which has a diameter with endpoints $P=(6.-3,-5)$ and $Q=(-2,5,-1)$. Find the center.

a. $(-4,4,2)$

b. $(4,-4,-2)$

c. $(2,1,-3)$ Correct

d. $(-2,-1,3)$

e. None of these

Solution: The center is the midpoint $M=\left(\frac{6-2}{2},\frac{-3+5}{2},\frac{-5-1}{2}\right)=(2,1,-3)$.

2. Consider the sphere which has a diameter with endpoints $P=(6.-3,-5)$ and $Q=(-2,5,-1)$. Find the radius.

a. 6 Correct

b. 12

c. 36

d. 144

e. None of these

Solution: The diameter is the distance between the endpoints:
$d=\sqrt{(-2-6)^2+(5--3)^2+(1-5)^2}=\sqrt{64+64+16}=\sqrt{144}=12$. So the radius is $r=6$

3. If $\vec{u}$ points Up and $\vec{v}$ points NorthWest, where does $\vec{u}\times\vec{v}$ point?

a. Down

b. SouthWest Correct

c. SouthEast

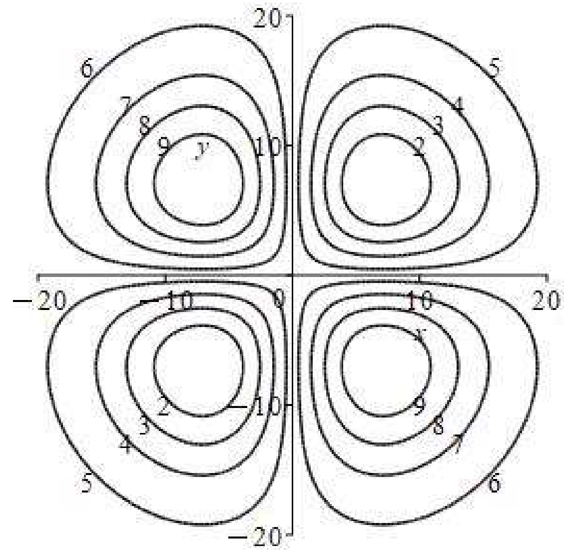
d. NorthEast

e. South

Solution: Hold your right hand with the fingers pointing Up and the palm facing NorthWest. Then the thumb points SouthWest.

4. The figure shows the contour plot of a function $f(x,y)$.
The level sets are labeled by the values of f.
Which point is a local maximum?

a. $(7,0)$
b. $(15,-15)$
c. $(-7,-7)$
d. $(-7,7)$ Correct
e. $(0,0)$



Solution: Contours circle around a maximum.
Looking at the values, one local maximum is at $(-7,7)$.

5. Consider the contour plot in the previous problem. Which point is a saddle point?

a. $(7,0)$
b. $(15,-15)$
c. $(-7,-7)$
d. $(-7,7)$
e. $(0,0)$ Correct

Solution: Contours bend away from a saddle point, as they do around $(0,0)$

6. Write the vector $\vec{u} = \langle 7,2,3 \rangle$ as the sum of a vector $\vec{p}$ which is parallel to $\vec{v} = \langle 3,2,1 \rangle$ and a vector $\vec{q}$ which is perpenducular to $\vec{v}$. Then $\vec{q} =$

a. $\langle 1,-2,1 \rangle$ Correct
b. $\langle -2,2,2 \rangle$
c. $\langle 2,-3,0 \rangle$
d. $\langle -2,4,-2 \rangle$
e. $\langle 1,-1,-1 \rangle$

Solution: $\vec{p} = \text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\vec{v} = \dfrac{21+4+3}{9+4+1}\langle 3,2,1 \rangle = 2\langle 3,2,1 \rangle = \langle 6,4,2 \rangle$

$\vec{q} = \vec{u} - \vec{p} = \langle 7,2,3 \rangle - \langle 6,4,2 \rangle = \langle 1,-2,1 \rangle$

7. Find a vector perpendicular to the plane containing the points

$$A = (2,0,4), \quad B = (2,1,3) \quad \text{and} \quad C = (3,1,4).$$

a. $\langle 1,1,-1\rangle$

b. $\langle -1,-1,-1\rangle$

c. $\langle 1,1,1\rangle$

d. $\langle -1,1,-1\rangle$

e. $\langle 1,-1,-1\rangle$ Correct

Solution: $\overrightarrow{AB} = B - A = \langle 0,1,-1\rangle \qquad \overrightarrow{AC} = C - A = \langle 1,1,0\rangle$

$$\vec{N} = \begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \hat{\imath}(0 - -1) - \hat{\jmath}(0 - -1) + \hat{k}(0-1) = \langle 1,-1,-1\rangle$$

8. Write $(4,1,-4)$ as a linear combination of $(2,3,3)$ and $(1,-1,-2)$ or determine that it is impossible.

In other words, find a and b so that

$$(4,1,-4) = a(2,3,3) + b(1,-1,-2)$$

Then $a + 2b =$

a. 3

b. 5

c. 8

d. 12

e. Impossible Correct

Solution: We need to solve $\begin{aligned} 2a + b &= 4 \\ 3a - b &= 1 \\ 3a - 2b &= -4 \end{aligned}$ Adding the first 2 equations gives $5a = 5$ or $a = 1$.

Substituting into the 1st equation gives $b = 2$. However, the left side of the 3rd equation gives $3a - 2b = 3(1) - 2(2) = -1$ which is not the right side. So there is NO SOLUTION.

9. Classify the surface: $2x^2 - 8x - y^2 + 6y + z^2 = 1$.

a. Hyperbolic Paraboloid

b. Hyperbolic Cylinder

c. Hyperboloid of 1 sheet

d. Hyperboloid of 2 sheets

e. Cone Correct

Solution: Complete the squares: $2(x^2 - 2x + 4) - (y^2 - 6x + 9) + z^2 = 1 + 8 - 9$

$2(x-2)^2 - (y-3)^2 + z^2 = 0$ Cone

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (15 points) Find the point where the line $(x,y,z) = (2-t, 1+2t, 3-t)$ intersects the plane $3x + 2y - 3z = 11$.

Solution: Substitute the line into the plane and solve for t:

$$3(2-t) + 2(1+2t) - 3(3-t) = 11 \quad \text{or} \quad -1 + 4t = 11 \quad \text{or} \quad t = 3$$

Substitute back into the line: $(x,y,z) = (2-3, 1+2\cdot 3, 3-3) = \boxed{(-1,7,0)}$

Check in the plane: $3\cdot(-1) + 2\cdot(7) - 3\cdot(0) = 11$

11. (25 points) For the curve $\vec{r}(t) = \langle e^{2t}, 2e^t, t\rangle$ compute each of the following:

a. (5 pts) The velocity $\vec{v}$

Solution: Differentiate the position: $\vec{v} = \underline{\langle 2e^{2t}, 2e^t, 1\rangle}$

b. (5 pts) The speed $\frac{ds}{dt}$ (Simplify!)

Solution: $\frac{ds}{dt} = |\vec{v}| = \sqrt{4e^{4t} + 4e^{2t} + 1} = \sqrt{(2e^{2t}+1)^2} = 2e^{2t} + 1$ $\qquad \frac{ds}{dt} = \underline{2e^{2t}+1}$

c. (5 pts) The tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(2e^{2t}+1) = 4e^{2t}$ $\qquad a_T = \underline{4e^{2t}}$

d. (5 pts) The length of this curve between $(1,2,0)$ and $(e^2, 2e, 1)$.

Solution: $|\vec{v}| = 2e^{2t} + 1 \qquad (1,2,0) = \vec{r}(0) \qquad (e^2, 2e, 1) = \vec{r}(1)$

$$L = \int_{(1,2,0)}^{(e^2,2e,1)} ds = \int_0^1 |\vec{v}|\,dt = \int_0^1 (2e^{2t}+1)\,dt = \left[e^{2t} + t\right]_0^1 = (e^2+1) - 1 = e^2$$

$L = \underline{e^2}$

e. (5 pts) The unit binormal vector $\hat{B}$

Solution: $\vec{v} = \langle 2e^{2t}, 2e^t, 1\rangle \qquad \vec{a} = \langle 4e^{2t}, 2e^t, 0\rangle$

$$\vec{v}\times\vec{a} = \begin{vmatrix} i & j & k \\ 2e^{2t} & 2e^t & 1 \\ 4e^{2t} & 2e^t & 0 \end{vmatrix} = \hat{\imath}(0 - 2e^t) - \hat{\jmath}(0 - 4e^{2t}) + \hat{k}(4e^{3t} - 8e^{3t}) = \langle -2e^t, 4e^{2t}, -4e^{3t}\rangle$$

$$|\vec{v}\times\vec{a}| = \sqrt{4e^{2t} + 16e^{4t} + 16e^{6t}} = 2e^t\sqrt{1 + 4e^{2t} + 4e^{4t}} = 2e^t(1+2e^{2t})$$

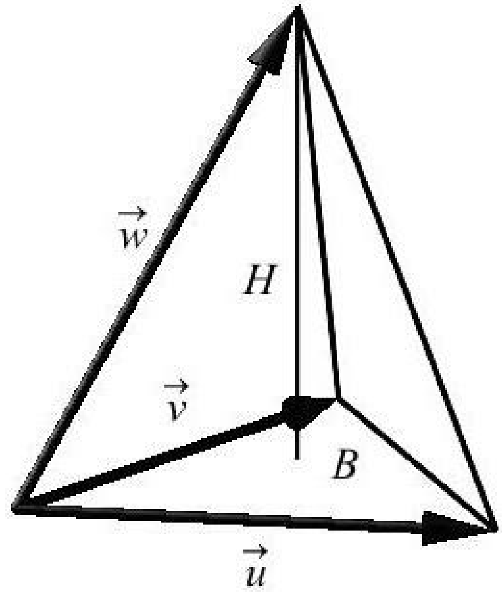
$$\hat{B} = \frac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|} = \frac{1}{2e^t(1+2e^{2t})}\langle -2e^t, 4e^{2t}, -4e^{3t}\rangle = \frac{1}{1+2e^{2t}}\langle -1, 2e^t, -2e^{2t}\rangle$$

$\hat{B} = \underline{\frac{1}{1+2e^{2t}}\langle -1, 2e^t, -2e^{2t}\rangle}$

12. (10 points) The volume of a pyramid is

$$V = \frac{1}{3}BH$$

where B is the area of the base and H is the height.
Derive a formula for the volume of the triangular pyramid with edge vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.
Your formula should involve the dot, cross and/or triple product of $\vec{u}$, $\vec{v}$ and $\vec{w}$.
Your derivation should explain all steps like the book does for the area of a triangle or the volume of a parallelepiped. Use sentences.



Solution: The area of the base triangle is

$$B = \frac{1}{2}|\vec{u} \times \vec{v}|.$$

Notice that $\vec{u} \times \vec{v}$ is perpendicular to the plane.
Let θ be the angle between $\vec{w}$ and $\vec{u} \times \vec{v}$.
Then there is a right triangle with hypotenuse $|\vec{w}|$ and adjacent side along $\vec{u} \times \vec{v}$ with length H.
Then

$$\cos\theta = \frac{H}{|\vec{w}|} \quad \text{or} \quad H = |\vec{w}|\cos\theta.$$

So the volume is

$$V = \frac{1}{3}BH = \frac{1}{3}\frac{1}{2}|\vec{u} \times \vec{v}||\vec{w}|\cos\theta = \frac{1}{6}\vec{u} \times \vec{v} \cdot \vec{w}$$

Since the volume must be positive but the triple product could be negative, the volume needs an absolute value:

$$V = \frac{1}{6}|\vec{u} \times \vec{v} \cdot \vec{w}|$$