Name________________________

1	/15	4	/20
2	/15	5	/20
3	/15	6	/20
		Total	/105

MATH 221 Exam 3 Spring 2023

Section 501 Solutions P. Yasskin

Work Out: (Points indicated. Part credit possible. Show all work.)

1. (15 points) Given the vector field $\vec{F}(x,y,z) = \langle xz^2, yz^2, z^3 \rangle$, compute the triple integral $\iiint \vec{\nabla} \cdot \vec{F}\, dV$ of its divergence over the solid between $y = x^2$ and $y = 2x$ for $0 \le z \le 3$.

Solution: The curves $y = x^2$ and $y = 2x$ intersect at $x = 0,2$. Between these, $x^2 < 2x$. The divergence of $\vec{F}$ is $\vec{\nabla} \cdot \vec{F} = z^2 + z^2 + 3z^2 = 5z^2$. So the integral is:

$$\iiint \vec{\nabla} \cdot \vec{F}\, dV = \int_0^2 \int_{x^2}^{2x} \int_0^3 5z^2\, dz\, dy\, dx = \int_0^3 5z^2\, dz \int_0^2 \int_{x^2}^{2x} 1\, dy\, dx = \left[5\frac{z^3}{3}\right]_0^3 \int_0^2 \Big[y\Big]_{x^2}^{2x} dx$$

$$= 45 \int_0^2 (2x - x^2)\, dx = 45\left[x^2 - \frac{x^3}{3}\right]_0^2 = 45\left(4 - \frac{8}{3}\right) = 60$$

2. (15 points) Given the function $f(x,y,z) = xy + 3z$ compute the vector line integral $\int_A^B \vec{\nabla} f \cdot d\vec{s}$ along the twisted cubic $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ between $A = \left(1, 1, \frac{2}{3}\right)$ and $B = (3, 9, 18)$.

Solution: The gradient is $\vec{\nabla} f = \langle y, x, 3 \rangle$. Along the curve, this is $\vec{\nabla} f\Big|_{\vec{r}(t)} = \langle t^2, t, 3 \rangle$.
The velocity is $\vec{v} = \langle 1, 2t, 2t^2 \rangle$. The endpoints are

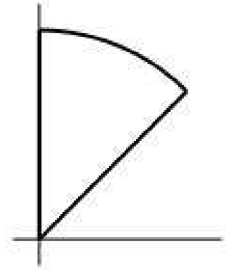
$$A = \left(1, 1, \frac{2}{3}\right) = \vec{r}(1) \qquad B = (3, 9, 18) = \vec{r}(3)$$

So the line integral is:

$$\int_A^B \vec{\nabla} f \cdot d\vec{s} = \int_1^3 \langle t^2, t, 3 \rangle \cdot \langle 1, 2t, 2t^2 \rangle\, dt = \int_1^3 (t^2 + 2t^2 + 6t^2)\, dt = \int_1^3 (9t^2)\, dt = \left[3t^3\right]_1^3 = 81 - 3 = 78$$

3. (15 points) Compute $\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$

Hint: Change coordinates.

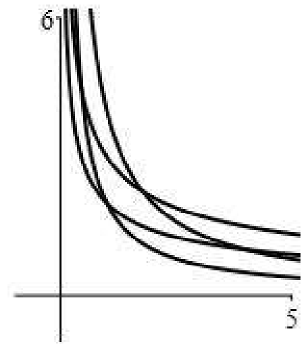


Solution: $\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx = \int_{\pi/4}^{\pi/2} \int_0^2 e^{-r^2} r\, dr\, d\theta = \left[\frac{\pi}{2} - \frac{\pi}{4} \right] \left[-\frac{1}{2} e^{-r^2} \right]_0^2 = \frac{\pi}{8}(1 - e^{-4})$

4. (20 points) Compute $\iint_D xy^2\, dA$ over the diamond shaped region in the first quadrant bounded by the curves

$$x = \frac{4}{y^2} \qquad x = \frac{9}{y^2} \qquad y = \frac{2}{x} \qquad y = \frac{4}{x}$$

HINT: Let $u = xy^2$ and $v = xy$. What are $\frac{v^2}{u}$ and $\frac{u}{v}$?



Solution: Let $u = xy^2$ and $v = xy$. Then the boundaries are:

$$u = xy^2 = 4 \qquad u = xy^2 = 9 \qquad v = xy = 2 \qquad v = xy = 4$$

Notice $\frac{v^2}{u} = \frac{x^2y^2}{xy^2} = x$ and $\frac{u}{v} = \frac{xy^2}{xy} = y$. So the position vector is

$$(x, y) = \vec{R}(u, v) = \left(\frac{v^2}{u}, \frac{u}{v} \right).$$

The Jacobian determinant is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{v^2}{u^2} & \frac{1}{v} \\ \frac{2v}{u} & -\frac{u}{v^2} \end{vmatrix} = \frac{1}{u} - \frac{2}{u} = \frac{-1}{u}$$

So the Jacobian factor is $J = \left| -\frac{1}{u} \right| = \frac{1}{u}$, and $dA = J\, du\, dv = \frac{1}{u}\, du\, dv$

The integrand is $xy^2 = \frac{v^2}{u} \left(\frac{u}{v} \right)^2 = u$. So the integral is

$$\iint_D xy^2\, dA = \int_2^4 \int_4^9 u \frac{1}{u}\, du\, dv = \int_2^4 \int_4^9 1\, du\, dv = \Big[v \Big]_2^4 \Big[u \Big]_4^9 = (2)(5) = 10$$

5. (20 points) Consider the solid cylinder $x^2+y^2 \le 4$ for $2 \le z \le 6$ with density is $\delta = (x^2+y^2)z$.

a. Find the mass of the cylinder.

Solution: In cylindrical coordinates, the density is $\delta = r^2 z$. So the mass is:

$$M = \iiint_C \delta\, dV = \int_2^6 \int_0^{2\pi} \int_0^2 r^2 z r\, dr\, d\theta\, dz = \left[\frac{z^2}{2}\right]_2^6 [2\pi] \left[\frac{r^4}{4}\right]_0^2 = \frac{36-4}{2}(2\pi)(4) = 128\pi$$

b. Find the center of mass of the cylinder.

Solution: By symmetry, $\bar{x} = \bar{y} = 0$. The z-moment is:

$$M_z = \iiint_C z\delta\, dV = \int_2^6 \int_0^{2\pi} \int_0^2 z r^2 z r\, dr\, d\theta\, dz = \left[\frac{z^3}{3}\right]_2^6 [2\pi] \left[\frac{r^4}{4}\right]_0^2 = \frac{216-8}{3}(2\pi)(4) = \frac{1664}{3}\pi$$

So the z component of the center of mass is $\bar{z} = \dfrac{M_z}{M} = \dfrac{1664\pi}{3}\dfrac{1}{128\pi} = \dfrac{13}{3}$

6. (20 points) Given the vector field $\vec{F}(x,y,z) = \langle yz^2, -xz^2, z^3 \rangle$ compute the vector surface integral $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ along the side surface of the cylinder $x^2+y^2 = 4$ for $2 \le z \le 6$, oriented outward. (There are no ends on the cylinder.) Parametrize the cylinder by $\vec{R}(z,\theta) = (2\cos\theta, 2\sin\theta, z)$.

Solution: The tangent vectors and normal vector to the surface are:

$$\begin{aligned} & \quad\ \ \hat{\imath} \qquad\ \ \hat{\jmath} \qquad \hat{k} \\ \vec{e}_z &= \langle\ 0, \qquad 0, \qquad 1 \rangle \\ \vec{e}_\theta &= \langle -2\sin\theta, 2\cos\theta, 0 \rangle \\ \vec{N} &= \langle -2\cos\theta, -2\sin\theta, 0 \rangle \end{aligned}$$

To have $\vec{N}$ oriented outward, we reverse it:

$$\vec{N} = \langle 2\cos\theta, 2\sin\theta, 0 \rangle.$$

The curl of $\vec{F}$ is:

$$\begin{aligned} \vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz^2 & -xz^2 & z^3 \end{vmatrix} \\ &= \hat{\imath}(2xz) - \hat{\jmath}(-2yz) + \hat{k}(-z^2 - z^2) \\ &= \langle 2xz, 2yz, -2z^2 \rangle \end{aligned}$$

On the surface, this is:

$$\left[\vec{\nabla} \times \vec{F}\right]_{\vec{R}(\theta,z)} = \langle 4z\cos\theta, 4z\sin\theta, -2z^2 \rangle$$

The necessary dot product is:

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = 8z\cos^2\theta + 8z\sin^2\theta + 0 = 8z$$

So the surface integral is:

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_2^6 \int_0^{2\pi} \vec{\nabla} \times \vec{F} \cdot \vec{N}\, d\theta\, dz = \int_2^6 \int_0^{2\pi} 8z\, d\theta\, dz = 2\pi\left[4z^2\right]_2^6 = 8\pi(36-4) = 256\pi$$