Name________________________

MATH 221 Final Spring 2023

Section 501 P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1-9	/36	12	/15
10	/15	13	/25
11	/15	Total	/106

1. Find the angle between the line $\vec{r}(t) = (3+2t, 2, 5+2t)$ and the normal to the plane $x+y+2z=4$.

a. $\frac{\pi}{6}$

b. $\frac{\pi}{4}$

c. $\frac{\pi}{3}$

d. $\frac{\pi}{2}$

e. $\frac{2\pi}{3}$

2. Find the equation of the plane tangent to $z = x^2y + y^2x$ at the point $(x,y) = (1,2)$.
Which of the following points lies on the tangent plane?

a. $(2,1,19)$

b. $(2,1,9)$

c. $(3,3,17)$

d. $(3,3,21)$

3. Find the plane tangent to the surface $x^2z + y^2z + xyz = 21$ at the point $P = (1, 2, 3)$.
Find the z-intercept.

a. $z = 3$

b. $z = 5$

c. $z = 7$

d. $z = 9$

e. $z = 11$

4. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. A cone currently has radius $r = 5$ cm and height $h = 8$ cm.
If the radius decreases at $0.3 \frac{\text{cm}}{\text{sec}}$ while the volume decreases by $8\pi \frac{\text{cm}^3}{\text{sec}}$,
find the rate at which the height is currently changing. $\frac{dh}{dt} =$

a. $\frac{3}{25} \frac{\text{cm}}{\text{sec}}$

b. $\frac{48}{25} \frac{\text{cm}}{\text{sec}}$

c. $-\frac{25}{3} \frac{\text{cm}}{\text{sec}}$

d. $-\frac{25}{48} \frac{\text{cm}}{\text{sec}}$

e. $0 \frac{\text{cm}}{\text{sec}}$

5. The function $f(x,y) = x^4 - 8xy + \frac{1}{16}y^4$ has a critical point at $(2,4)$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum

b. Local Maximum

c. Inflection Point

d. Saddle Point

e. Test Fails

6. Compute $\int_0^8 \int_{x^{1/3}}^2 \cos(y^4)\, dy\, dx$

HINT: Reverse the order of integration.

a. $\frac{1}{4}\sin(4) - \frac{1}{4}$

b. $\frac{1}{4}\sin(64) - \frac{1}{4}$

c. $\frac{1}{4}\sin(64)$

d. $\frac{1}{4}\sin(16) - \frac{1}{4}$

e. $\frac{1}{4}\sin(16)$

7. Consider the parametric surface $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

Find the normal line at the point $P = \vec{R}\left(\sqrt{2}, \frac{\pi}{4}\right) = (1,1,2)$.

It intersects the xy-plane at

a. $(-3,-3,0)$

b. $(-3,-3,4)$

c. $(5,5,0)$

d. $(5,5,4)$

e. $\left(2\sqrt{2}, 2\sqrt{2}, 0\right)$

8. On Exam 3, you solved the problem:

"Given the function $f(x,y,z) = xy + 3z$ compute the vector line integral $\int_A^B \vec{\nabla} f \cdot d\vec{s}$ along the twisted cubic $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ between $A = \left(1, 1, \frac{2}{3}\right)$ and $B = (3,9,18)$."

You can now do it more easily using a Theorem. Which Theorem?

a. Fundamental Theorem of Calculus for Curves

b. Green's Theoprem

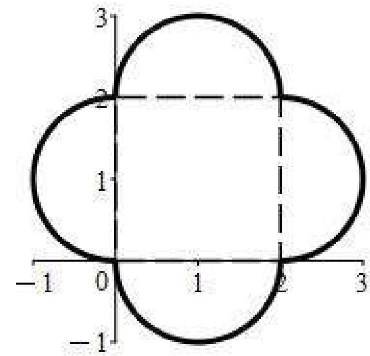
c. 2D Stokes' Theorem

d. Stokes' Theorem

e. Gauss' Theorem

9. Compute the line integral $\oint (3y + \cos x)\,dx + (5x - \sin y)\,dy$ counterclockwise around the boundary of the region shown consisting of a square and 4 semicircles.

HINT: Use a Theorem.



a. $4 + 2\pi$

b. $1 + 2\pi$

c. $8 + 4\pi$

d. $\pi + 2\pi^2$

e. $2\pi + 4\pi^2$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. $\left(15 \text{ points}\right)$ Find the volume of the largest rectangular solid with 3 faces in the coordinate planes and the opposite vertex on the plane $\frac{x}{9} + \frac{y}{6} + \frac{z}{3} = 1$.

11. $\left(15 \text{ points}\right)$ Consider the parametric surface $\vec{R}(u,v) = \left(u^2, v^2, \sqrt{2}\,uv\right)$

for $0 \le u \le 2$ and $0 \le v \le 3$.

Find the mass of the surface if the surface density is $\delta = \frac{1}{x+y}$.

HINT: Factor out a $\sqrt{8}$.

12. (15 points) Given the vector field $\vec{F}(x,y,z) = \langle yz^2, -xz^2, z^3 \rangle$ compute the vector surface integral $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ along the side surface of the cylinder $x^2 + y^2 = 4$ for $2 \le z \le 6$, oriented **outward**. (There are no ends on the cylinder.) On Exam 3, you solved this directly.
Now solve it using Stokes' Theorem, using the following steps.

a. Compute the line integral $\int_{z=6} \vec{F} \cdot d\vec{s}$ around the circle $x^2 + y^2 = 4$ for $z = 6$, **counterclockwise** as seen from above.
The circle may be parametrized by $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 6)$.

The velocity is $\vec{v} =$

On the circle $\vec{F}\Big|_{\vec{r}(\theta)} =$

$\int_{z=6} \vec{F} \cdot d\vec{s} =$

b. Compute the line integral $\int_{z=6} \vec{F} \cdot d\vec{s}$ around the circle $x^2 + y^2 = 4$ for $z = 2$, **counterclockwise** as seen from above.
The circle may be parametrized by $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2)$.

The velocity is $\vec{v} =$

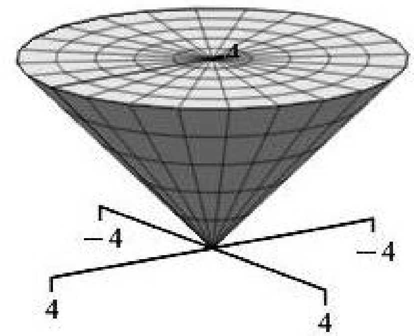
On the circle $\vec{F}\Big|_{\vec{r}(\theta)} =$

$\int_{z=2} \vec{F} \cdot d\vec{s} =$

c. Combine the answers to parts (a) and (b) (justifying your orientations) to find

$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} =$

13. (25 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = \langle yz^2, xz^2, z(x^2 + y^2) \rangle$ and the solid cone $\sqrt{x^2 + y^2} \le z \le 4$



Be sure to check orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence of $\vec{F}$:

$\vec{\nabla} \cdot \vec{F} =$

b. Compute the left hand side:

$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV =$

Second the Right Hand Side: The boundary surface consists of a disk and a cone.

Disk:

c. Parametrize the disk.

$\vec{R}(r,\theta) =$

d. Compute the tangent vectors:

$\vec{e}_r =$

$\vec{e}_\theta =$

e. Compute the normal vector:

$\vec{N} =$

f. Evaluate $\vec{F} = \langle yz^2, xz^2, z(x^2 + y^2) \rangle$ on the disk:

$\vec{F}\big|_{\vec{R}(r,\theta)} =$

g. Compute the dot product:

$\vec{F} \cdot \vec{N} =$

h. Compute the flux through D:

$\iint_D \vec{F} \cdot d\vec{S} =$

(continued)

Cone:

The cone may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

i. Compute the tangent vectors:

$\vec{e}_r =$

$\vec{e}_\theta =$

j. Compute the normal vector:

$\vec{N} =$

k. Evaluate $\vec{F} = \langle yz^2, xz^2, z(x^2+y^2)\rangle$ on the cone:

$\vec{F}\big|_{\vec{R}(r,\theta)} =$

l. Compute the dot product:

$\vec{F} \cdot \vec{N} =$

m. Compute the flux through C:

$\iint_C \vec{F} \cdot d\vec{S} =$

n. Compute the **TOTAL** right hand side:

Solution: $\iint_{\partial V} \vec{F} \cdot d\vec{S} =$