Name________________________________ Section:__________

MATH 221 Exam 3, Version A Fall 2023

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Multiple Choice: (6 points each. No part credit.)

1-8	/48	10	/16
9	/16	11	/24
		Total	/104

1. Find the divergence of the vector field $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$ and evaluate the divergence at $P = (1,2,3)$.

a. -2

b. 20

c. 22 Correct

d. 23

e. 24

Solution: $\vec{\nabla} \cdot F = 2xy + 2yz + 2zx$ $\quad \vec{\nabla} \cdot F\Big|_P = 4 + 12 + 6 = 22$

2. Find the curl of the vector field $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$ and evaluate the curl at $P = (1,2,3)$.

a. $\langle -4, -9, -1 \rangle$ Correct

b. $\langle -4, 9, -1 \rangle$

c. $\langle 2, -12, 9 \rangle$

d. $\langle 2, 12, 9 \rangle$

e. $\langle 1, -4, 9 \rangle$

Solution: $\vec{\nabla} \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & y^2z & z^2x \end{vmatrix} = \hat{\imath}(-y^2) - \hat{\jmath}(z^2) + \hat{k}(-x^2)$ $\quad \vec{\nabla} \times F\Big|_P = \langle -4, -9, -1 \rangle$

3. Let f be a scalar potential for $\vec{F} = \langle yz + 2x, xz + 2y, xy + 2z \rangle$. Compute $f(1,2,3) - f(0,0,0)$.
(Note: The subtraction cancels off the arbitrary constant.)

a. -2

b. 20 Correct

c. 22

d. 23

e. 24

Solution: By inspection, $f = xyz + x^2 + y^2 + z^2$.
Then $f(1,2,3) - f(0,0,0) = (6 + 1 + 4 + 9) - (0) = 20$

4. Use a Riemann sum with 6 squares evaluated at the center of each square to estimate the volume of the solid over the rectangle $[1,7]\times[2,6]$ below the surface $f = x^2 + y^2$.

a. 130

b. 214

c. 520

d. 856 Correct

e. 872

Solution: The volume is $V = \iint f\,dA$. The area of each square is $\Delta A = \Delta x\,\Delta y = 2\cdot 2 = 4$.
The centers are:

$$(2,3),\quad (2,5),\quad (4,3),\quad (4,5),\quad (6,3),\quad (6,5)$$

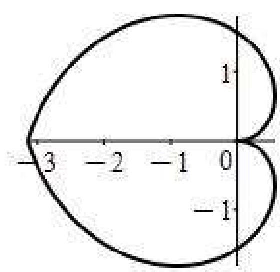
The function values are:

$$f(2,3) = 13,\quad f(2,5) = 29,\quad f(4,3) = 25,\quad f(4,5) = 41,\quad f(6,3) = 45,\quad f(6,5) = 61$$

The Riemann sum is:

$$V = \iint f\,dA \approx \sum_{k=1}^{6} f(x_i^*, y_j^*)\,\Delta A = (13+29+25+41+45+61)4 = 856$$

5. Find the area inside the heart which in polar coordinates is the spiral $r = |\theta|$ for $-\pi \le \theta \le \pi$.
HINT Double the area inside half the spiral.



a. $\frac{\pi^3}{6}$

b. $\frac{\pi^3}{3}$ Correct

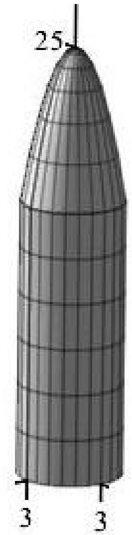
c. $\frac{\pi^3}{2}$

d. $\frac{\pi^4}{4}$

e. $\frac{\pi^4}{2}$

Solution: $A = \iint 1\,dA = 2\int_0^{\pi}\int_0^{\theta} r\,dr\,d\theta = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\theta} d\theta = \int_0^{\pi}\theta^2\,d\theta = \left[\frac{\theta^3}{3}\right]_0^{\pi} = \frac{\pi^3}{3}$

6. Find mass of the solid below $z = 25 - x^2 - y^2$
above the xy-plane inside the cylinder $x^2 + y^2 = 9$
if the volume density is $\delta = z$.



a. $\frac{\pi}{6}(25^3 - 16^3)$ Correct

b. $\frac{\pi}{6}(25^3 + 16^3)$

c. $\pi\left(25^2 \cdot 3 - 50 \cdot 3^2 + \frac{3^5}{5}\right)$

d. $\pi\left(25^2 \cdot 3 + 50 \cdot 3^2 + \frac{3^5}{5}\right)$

e. $\pi\left(25\frac{3^2}{2} - \frac{3^4}{4}\right)$

Solution: In cylindrical coordinates, the top is $z = 25 - r^2$, the sides are $r = 3$ and the volume differential is $dV = r\,dr\,d\theta\,dz$.

$$M = \iiint_V \delta\,dV = \int_0^{2\pi}\int_0^3\int_0^{25-r^2} z\,r\,dz\,dr\,d\theta = 2\pi\int_0^3\left[\frac{z^2}{2}\right]_0^{25-r^2} r\,dr = \pi\int_0^3 (25-r^2)^2\,r\,dr$$
$$= -\frac{\pi}{2}\int_{25}^{16} u^2\,du = -\frac{\pi}{2}\left[\frac{u^3}{3}\right]_{25}^{16} = -\frac{\pi}{6}(16^3 - 25^3) = \frac{\pi}{6}(25^3 - 16^3)$$

7. Find the Jacobian factor for the 3D coordinate system:

$$(x,y,z) = \vec{R}(u,v,w) = (vw, uw, uv) \qquad \text{with} \quad u > 0, \quad v > 0, \quad w > 0$$

a. uvw

b. $2uvw$ Correct

c. $u + v + w$

d. $2u + 2v + 2w$

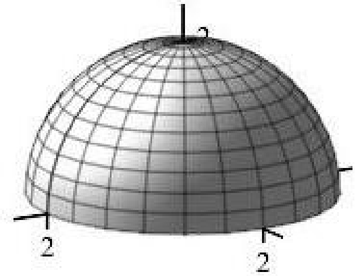
e. $vw + uw + uv$

Solution: The Jacobian determinant is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{vmatrix} = 0|*| - w\begin{vmatrix} w & u \\ v & 0 \end{vmatrix} + v\begin{vmatrix} w & 0 \\ v & u \end{vmatrix} = -w(-uv) + v(wu) = 2uvw$$

Since this is positive $J = 2uvw$.

8. Find the average value of the function $f = x^2 + y^2 + z^2$ on the solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$.



a. $\frac{5}{5}$

b. $\frac{4}{5}\pi$

c. $\frac{12}{5}$ Correct

d. $\frac{16}{3}\pi$

e. $\frac{64}{5}\pi$

Solution: The radius of the sphere is $\rho = 2$. The volume of the hemisphere is $V = \frac{2}{3}\pi\rho^3 = \frac{16}{3}\pi$.
In spherical coordinates the function is $f = \rho^2$. So the integral of the function is:

$$\iiint f\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \Big[\theta\Big]_0^{2\pi}\Big[-\cos\phi\Big]_0^{\pi/2}\Big[\frac{\rho^5}{5}\Big]_0^2 = (2\pi)(1)\frac{32}{5} = \frac{64\pi}{5}$$

So the average of f is $f_{\text{ave}} = \frac{1}{V}\iiint f\,dV = \frac{3}{16\pi}\frac{64\pi}{5} = \frac{12}{5}$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (16 points) Consider a plate bounded by $y = 4 - x^2$ and the x-axis with surface density $\delta = x^2$.

a. (8 pts) Find the mass of a plate.

Solution: The mass is:

$$M = \iint \delta\,dA = \int_{-2}^{2}\int_0^{4-x^2} x^2\,dy\,dx = \int_{-2}^{2}\Big[x^2 y\Big]_{y=0}^{4-x^2} dx = \int_{-2}^{2} x^2(4-x^2)\,dx$$
$$= \Big[\frac{4x^3}{3} - \frac{x^5}{5}\Big]_{-2}^{2} = 2\Big(\frac{32}{3} - \frac{32}{5}\Big) = 64\frac{2}{15} = \frac{128}{15}$$

b. (8 pts) Find the center of mass of a plate.

Solution: By symmetry $\bar{x} = 0$. The y-moment is:

$$M_y = \iint y\,\delta\,dA = \int_{-2}^{2}\int_0^{4-x^2} yx^2\,dy\,dx = 3\int_{-2}^{2}\Big[x^2\frac{y^2}{2}\Big]_{y=0}^{4-x^2} dx = \frac{1}{2}\int_{-2}^{2} x^2(4-x^2)^2\,dx$$
$$= \frac{1}{2}\int_{-2}^{2} x^2(16 - 8x^2 + x^4)\,dx = \frac{1}{2}\Big[16\frac{x^3}{3} - 8\frac{x^5}{5} + \frac{x^7}{7}\Big]_{-2}^{2} = \Big(\frac{2^7}{3} - \frac{2^8}{5} + \frac{2^7}{7}\Big)$$
$$= 2^7\Big(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\Big) = \frac{1024}{105}$$

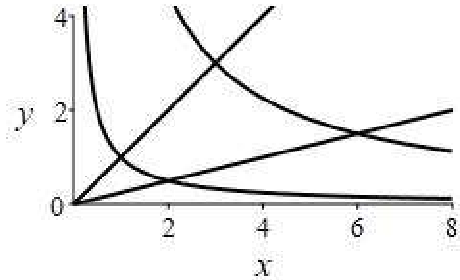
So $\bar{y} = \frac{M_y}{M} = \frac{1024}{105}\frac{15}{128} = \frac{8}{7}$

10. (16 points) Compute $\iint_D x^2\,dA$ over the diamond shaped region in the 1^{st} quadrant bounded by

$$y = \frac{1}{x} \qquad y = \frac{9}{x} \qquad y = x \qquad y = \frac{1}{4}x$$

HINT: Use the curvilinear coordinates

$$x = uv \qquad y = \frac{v}{u}.$$



a. (5 pts) Find the Jacobian factor.

Solution: $J = \left| \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \begin{vmatrix} v & -\dfrac{v}{u^2} \\ u & \dfrac{1}{u} \end{vmatrix} \right| = \left| \frac{v}{u} - -\frac{v}{u} \right| = \frac{2v}{u}$

b. (1 pts) Express the integrand in terms of the coordinates.

Solution: $x^2 = u^2 v^2$.

c. (4 pts) Substitute $x = uv$ and $y = \frac{v}{u}$ into the boundaries to express them in terms of u and v.

Solution: $xy = uv\frac{v}{u} = v^2$ So two boundaries are $v^2 = 1$ and $v^2 = 9$ or $v = 1$ and $v = 3$.
$\frac{y}{x} = \frac{v}{u}\frac{1}{uv} = \frac{1}{u^2}$. So two boundaries are $\frac{1}{u^2} = 1$ and $\frac{1}{u^2} = \frac{1}{4}$ or $u = 1$ and $u = 2$.

d. (6 pts) Compute the integral.

Solution:
$$\iint_D x^2\,dA = \int_1^3 \int_1^2 u^2 v^2 \cdot \frac{2v}{u}\,du\,dv = \int_1^3 2v^3\,dv \int_1^2 u\,du = \left[\frac{v^4}{2}\right]_1^3 \left[\frac{u^2}{2}\right]_1^2 = \frac{81-1}{2}\,\frac{4-1}{2} = 60$$

11. (24 points) Consider the cone surface $z = 2\sqrt{x^2 + y^2}$ for $z \le 6$ which may be parametrized by

$$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle$$

a. (6 pts) Find the tangent vectors:

Solution:

$$\begin{array}{rl} & \quad\hat{\imath} \qquad\quad \hat{\jmath} \qquad \hat{k} \\ \vec{e}_r = & \Big\langle\ \cos\theta, \quad \sin\theta,\ 2\Big\rangle \\ \vec{e}_\theta = & \langle -r\sin\theta, r\cos\theta, \quad 0\rangle \end{array}$$

b. (3 pts) Find the normal vector oriented down and out:

Solution: $\vec{N} = \hat{\imath}(-2r\cos\theta) - \hat{\jmath}(2r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = \langle -2r\cos\theta, -2r\sin\theta, r\rangle$
This is up and in. So we reverse it: $\vec{N} = \langle 2r\cos\theta, 2r\sin\theta, -r\rangle$

c. (2 pts) Find the length of the normal vector:

Solution: $\left|\vec{N}\right| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5}\,r$

d. (4 pts) Find the mass of the cone if the mass density is $\delta = \sqrt{x^2 + y^2}$.

Solution: $M = \iint \delta\, dS = \iint \delta \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^3 r\sqrt{5}\, r\, dr\, d\theta = 2\pi\sqrt{5}\left[\frac{r^3}{3}\right]_0^3 = 18\sqrt{5}\,\pi$

e. (3 pts) Find the **curl** of the vector field $\vec{F} = \langle yz, -xz, z^2\rangle$ in rectangular coordinates:

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & z^2 \end{vmatrix} = \hat{\imath}(0 + x) - \hat{\jmath}(0 - y) + \hat{k}(-z - z) = \langle x, y, -2z\rangle$

f. (2 pts) Evaluate the **curl** of $\vec{F}$ on the cone:

Solution: $\vec{\nabla} \times \vec{F}\Big|_{\vec{R}} = \langle x, y, -2z\rangle = \langle r\cos\theta, r\sin\theta, -4r\rangle$

g. (4 pts) Find the flux of the **curl** of $\vec{F}$ down and out of the cone:

Solution: $\vec{\nabla} \times \vec{F}\Big|_{\vec{R}} \cdot \vec{N} = 2r^2\cos^2\theta + 2r^2\sin^2\theta + 4r^2 = 6r^2 \qquad z = 2r \le 6 \qquad r \le 3$

$$\iint_C \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 6r^2\, dr\, d\theta = 2\pi\left[2r^3\right]_0^3 = 108\pi$$