Name______________________ Section:________

MATH 221 Final Exam, Version A Fall 2023

502,503 Solutions P. Yasskin

1-12	/60	14	/10
13	/10	15	/24
		Total	/104

Multiple Choice: (5 points each. No part credit.)

1. Consider the triangle with vertices $A = (2,4)$, $B = (1,1)$ and $C = (0,3)$.
Find the angle at B.

a. $30°$

b. $45°$ Correct

c. $60°$

d. $120°$

e. $135°$

Solution: $\overrightarrow{BA} = A - B = (1,3) \qquad \overrightarrow{BC} = C - B = (-1,2)$

$$\left|\overrightarrow{BA}\right| = \sqrt{1^2 + 3^2} = \sqrt{10} \qquad \left|\overrightarrow{BC}\right| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad \overrightarrow{BA} \cdot \overrightarrow{BC} = -1 + 6 = 5$$

$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}} \qquad \theta = 45°$$

2. Find the arc length of the curve $\vec{r}(t) = \langle e^t, 2t, 2e^{-t} \rangle$ between $(1,0,2)$ and $(e,2,2e^{-1})$.
Hint: Look for a perfect square.

a. $e + 2e^{-1}$

b. $e + 2e^{-1} - 3$

c. $e - 2e^{-1}$

d. $e - 2e^{-1} + 1$ Correct

e. $e - 2e^{-1} - 1$

Solution: $\vec{v} = \langle e^t, 2, -2e^{-t} \rangle \qquad |\vec{v}| = \sqrt{e^{2t} + 2 + 4e^{-2t}} = \sqrt{(e^t + 2e^{-t})^2} = e^t + 2e^{-t}$

$$L = \int ds = \int_0^1 |\vec{v}|\,dt = \int_0^1 (e^t + 2e^{-t})\,dt = [e^t - 2e^{-t}]_0^1 = (e - 2e^{-1}) - (1 - 2) = e - 2e^{-1} + 1$$

3. Find the point where the lines $(x,y,z) = (3-t, 2+t, 2t)$ and $(x,y,z) = (-1+2t, 5-t, 3+t)$ intersect. At this point $x+y+z =$

a. $\frac{17}{2}$

b. 9 Correct

c. $\frac{25}{3}$

d. 11

e. They do not intersect.

Solution: Change the name of the parameter in the second equation: $(x,y,z) = (-1+2s, 5-s, 3+s)$
Equate x, y and z components: $3-t = -1+2s,\quad 2+t = 5-s,\quad 2t = 3+s$
Add the first 2 equations: $5 = 4+s$ So $s = 1$
Plug into the 2nd equation: $2+t = 5-1 = 4$ So $t = 2$
Check the 3rd equation is satisfied: $2t = 4$ and $3+s = 4$ OK
Plug $t = 2$ into 1st line: $(x,y,z) = (3-t, 2+t, 2t) = (1,4,4)$
Plug $s = 1$ into 2nd line: $(x,y,z) = (-1+2s, 5-s, 3+s) = (1,4,4)$ So $x+y+z = 9$

4. Find the equation of the plane tangent to the graph of $f(x,y) = x^2y + xy^2$ at the point $(2,1)$. Its z-intercept is

a. -12 Correct

b. -6

c. 0

d. 6

e. 12

Solution: $f(x,y) = x^2y + xy^2 \qquad f_x(x,y) = 2xy + y^2 \qquad f_y(x,y) = x^2 + 2xy$
$f(2,1) = 6 \qquad f_x(2,1) = 5 \qquad f_y(2,1) = 8$
$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 6 + 5(x-2) + 8(y-1)$
z-intercept $= 6 - 10 - 8 = -12$

5. Find the plane tangent to the surface $x^2z^2 + y^4 = 5$ at the point $(2,1,1)$.

a. $2x + y + z = 6$

b. $2x + y + z = 5$

c. $x + y + 2z = 5$ Correct

d. $x - y + 2z = 3$

e. $x - y + 2z = 6$

Solution: $f = x^2z^2 + y^4 \qquad P = (2,1,1) \qquad \vec{\nabla} f = (2xz^2, 4y^3, 2x^2z) \qquad \vec{N} = \vec{\nabla} f\big|_P = (4,4,8)$
$\vec{N} \cdot X = \vec{N} \cdot P \qquad 4x + 4y + 8z = 8 + 4 + 8 = 20 \qquad x + y + 2z = 5$

6. The dimensions of a closed rectangular box are measured as 70 cm, 50 cm and 40 cm with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in the calculated surface area of the box.

a. 8

b. 16

c. 32

d. 64

e. 128 Correct

Solution: $A = 2xy + 2xz + 2yz$

$dA = \frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial y}\,dy + \frac{\partial A}{\partial z}\,dz = 2(y+z)\,dx + 2(x+z)\,dy + 2(x+y)\,dz$

$dA = 2(90)(.2) + 2(110)(.2) + 2(120)(.2) = .4(320) = 128$

7. Compute the line integral $\int -y\,dx + x\,dy$ **clockwise** around the semicircle $x^2 + y^2 = 9$ from $(-3,0)$ to $(3,0)$.
HINT: Parametrize the curve.

a. -9π Correct

b. -3π

c. π

d. 3π

e. 9π

Solution: $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta)$ $\quad \vec{v} = (-3\sin\theta, 3\cos\theta)$
For clockwise, reverse $\vec{v} = (3\sin\theta, -3\cos\theta)$.
$\vec{F} = (-y,x) = (-3\sin\theta, 3\cos\theta)$ $\quad \vec{F}\cdot\vec{v} = -9\sin^2\theta - 9\cos^2\theta = -9$
$\int y\,dx - x\,dy = \int \vec{F}\cdot d\vec{s} = \int \vec{F}\cdot\vec{v}\,d\theta = \int_0^\pi -9\,d\theta = -9\pi$

8. Compute $\int \vec{F}\cdot d\vec{s}$ for $\vec{F} = (y,x)$ along the curve $\vec{r}(t) = \left(e^{\cos(t^2)}, e^{\sin(t^2)}\right)$ for $0 \le t \le \sqrt{\pi}$.
HINT: Find a scalar potential.

a. $e - \frac{1}{e}$

b. $\frac{1}{e} - e$ Correct

c. $\frac{2}{e}$

d. $2e$

e. 0

Solution:
$\vec{F} = \vec{\nabla} f$ for $f = xy$ $\quad A = \vec{r}(0) = (e^{\cos 0}, e^{\sin 0}) = (e,1)$ $\quad B = \vec{r}(\sqrt{\pi}) = (e^{\cos\pi}, e^{\sin\pi}) = (e^{-1},1)$
By the FTCC: $\int_A^B \vec{F}\cdot d\vec{s} = \int_A^B \vec{\nabla} f\cdot d\vec{s} = f(B) - f(A) = f(e^{-1},1) - f(e,1) = (e^{-1}\cdot 1) - (e\cdot 1) = \frac{1}{e} - e$

9. Compute $\oint_S \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-xy^2 + x^3, x^2y - y^3)$ counterclockwise around the square $[0,3] \times [0,2]$.
Hint: Use a theorem.

a. 6

b. 12

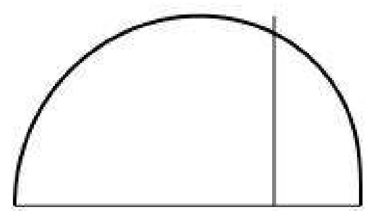
c. 16

d. 24

e. 36 Correct

Solution: We identify $P = -xy^2 + x^3$ and $Q = x^2y - y^3$. So $\partial_x Q - \partial_y P = 2xy - -2xy = 4xy$.
By Green's Theorem: $\int_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S (\partial_x Q - \partial_y P)\, dx\, dy = \int_0^2 \int_0^3 4xy\, dx\, dy = [x^2]_0^3 [y^2]_0^2 = 36$

10. Find the mass of the region inside the upper half of the limaçon $r = 2 - \cos\theta$ if the surface density is $\delta = y$.



a. $\frac{20}{3}$ Correct

b. $\frac{15}{3}$

c. $\frac{13}{3}$

d. $\frac{10}{3}$

e. $\frac{5}{3}$

Solution:

$$M = \iint \delta\, dA = \iint y\, dA = \int_0^{\pi} \int_0^{2-\cos\theta} r\sin\theta\, r\, dr\, d\theta = \int_0^{\pi} \left[\frac{r^3}{3} \right]_{r=0}^{2-\cos\theta} \sin\theta\, d\theta = \frac{1}{3} \int_0^{\pi} (2 - \cos\theta)^3 \sin\theta\, d\theta$$

$u = 2 - \cos\theta \qquad du = \sin\theta\, d\theta$

$$M = \frac{1}{3} \int_1^3 u^3\, du = \left[\frac{u^4}{12} \right]_1^3 = \frac{81 - 1}{12} = \frac{20}{3}$$

11. Consider the vector field $\vec{F} = \vec{\nabla} \times \vec{G}$ where $\vec{G} = \langle x^2 z, y^2 z, x^3 + y^3 \rangle$.
In which quadrant is $\vec{F}$ **always diverging**?

a. I

b. II

c. III

d. IV

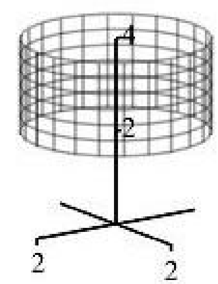
e. None of them   Correct

Solution: Since $\vec{\nabla} \cdot \vec{\nabla} \times \vec{G} = 0$ for any $\vec{G}$, we know $\vec{\nabla} \cdot \vec{F} = 0$ everywhere. $\vec{F}$ never diverges.

12. Consider the cylinder C given by $x^2 + y^2 = 4$
for $2 \le z \le 4$ with normal pointing outward.
Let T be the top circle and B be the bottom circle
both oriented counterclockwise as seen from above.
For a certain vector field $\vec{F}$ we have:

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = 14 \quad \text{and} \quad \oint_B \vec{F} \cdot d\vec{s} = 3$$

Compute $\oint_T \vec{F} \cdot d\vec{s}$.



a. 17

b. 11

c. 8

d. -11 Correct

e. -17

Solution: By Stokes' Theorem: $\iint_C \vec{\nabla} \times \vec{F}\, dV = -\oint_T \vec{F} \cdot d\vec{s} + \oint_B \vec{F} \cdot d\vec{s}$

The minus sign reverses the orientation of T to point clockwise. Thus

$$\oint_T \vec{F} \cdot d\vec{s} = \oint_B \vec{F} \cdot d\vec{s} - \iint_C \vec{\nabla} \times \vec{F}\, dV = 3 - 14 = -11$$

Work Out: (Points shown. Part credit possible. Show all work.)

13. (10 points) Find the dimensions and volume of the largest box which sits on the xy-plane and whose upper vertices are on the elliptic paraboloid $z + 2x^2 + 3y^2 = 12$.

You do not need to show it is a maximum.

Solution: We maximize $V = LWH = (2x)(2y)z = 4xyz$ subject to the constraint $g = z + 2x^2 + 3y^2 = 12$.

$\vec{\nabla} V = (4yz, 4xz, 4xy) \qquad \vec{\nabla} g = (4x, 6y, 1)$

$\vec{\nabla} V = \lambda \vec{\nabla} g \qquad \Rightarrow \qquad 4yz = \lambda 4x \qquad 4xz = \lambda 6y \qquad 4xy = \lambda$

$\lambda = 4xy \qquad \Rightarrow \qquad 4yz = 16x^2 y \qquad 4xz = 24xy^2$

Since $V \neq 0$, we can assume $x \neq 0$ and $y \neq 0$ and $z \neq 0$.

So $z = 4x^2 \qquad z = 6y^2 \qquad 2x^2 = 3y^2$

The constraint becomes: $4x^2 + 2x^2 + 2x^2 = 12$ or $8x^2 = 12$

$x = \sqrt{\frac{3}{2}} \qquad y = \sqrt{\frac{2}{3}}\, x = 1 \qquad z = 4x^2 = 6$

The dimensions are: $L = 2x = \sqrt{6} \qquad W = 2y = 2 \qquad H = z = 6$

The volume is: $V = LWH = \sqrt{6}\,(2)(6) = 12\sqrt{6}$

14. (10 points) Find the mass and center of mass of the conical **surface** $z = \sqrt{x^2 + y^2}$ for $z \le 2$ with density $\delta = x^2 + y^2$. The cone may be parametrize as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.

Solution:

$$\begin{array}{rccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 1) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$$

$$\left|\vec{N}\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{2} \qquad \delta = r^2$$

$$M = \iint \delta\, dS = \int\int r^2 \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^2 r^3\sqrt{2}\, dr\, d\theta = 2\pi\sqrt{2}\left[\frac{r^4}{4}\right]_0^2 = 8\pi\sqrt{2}$$

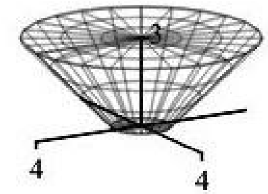
$\bar{x} = \bar{y} = 0$ by symmetry.

$$z\text{-mom} = M_z = \iint z\,\delta\, dS = \int\int r^3 \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^2 r^4\sqrt{2}\, dr\, d\theta = 2\pi\sqrt{2}\left[\frac{r^5}{5}\right]_0^2 = \frac{64\pi\sqrt{2}}{5}$$

$$\bar{z} = \frac{M_z}{M} = \frac{64\pi\sqrt{2}}{5}\,\frac{1}{8\pi\sqrt{2}} = \frac{8}{5}$$

15. (24 points) Verify Gauss' Theorem

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$$



for the vector field $\vec{F} = \langle x, y, 2z \rangle$ and the solid bowl filled with water bounded by the surfaces

$z = 0$ and $z = 3$ and $r = z + 1$.

Be sure to check orientations. Use the following steps:

Left Hand Side:

a. Compute the divergence of $\vec{F}$:

Solution: $\vec{\nabla} \cdot \vec{F} = 1 + 1 + 2 = 4$

b. Compute the left hand side: (Be careful with the bounds on r and z.)

Solution: $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^3\int_0^{z+1} 4r\, dr\, dz\, d\theta = 2\pi \int_0^3 \left[2r^2\right]_{r=0}^{z+1} dz = 4\pi \int_0^3 (z+1)^2\, dz$

$= 4\pi \left[\frac{(z+1)^3}{3}\right]_0^3 = 4\pi\left(\frac{64}{3} - \frac{1}{3}\right) = 84\pi$

Right Hand Side: The boundary surface consists of two disks and the side surface.

Side Surface S:

The sides may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r-1)$

c. Compute the normal vector and check its orientation:

Solution:

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r & = & (\ \cos\theta, & \sin\theta, & 1) \\ \vec{e}_\theta & = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(- -r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = \langle -r\cos\theta, -r\sin\theta, r\rangle$

This is up and in. We need down and out. So we reverse it: $\vec{N} = \langle r\cos\theta, r\sin\theta, -r\rangle$

d. Evaluate $\vec{F} = \langle x, y, 2z \rangle$ on the sides:

Solution: $\vec{F}\big|_{\vec{R}(r,\theta)} = \langle r\cos\theta, r\sin\theta, 2r-2\rangle$

e. Compute their dot product:

Solution: $\vec{F} \cdot \vec{N} = r^2\cos^2\theta + r^2\sin^2\theta - r(2r-2) = r^2 - 2r^2 + 2r = -r^2 + 2r$

f. Compute the flux through S: (Be careful with the bounds on r.)

Solution: $\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_1^4 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_1^4 (-r^2 + 2r)\, dr\, d\theta$

$= 2\pi\left[-\frac{r^3}{3} + r^2\right]_1^4 = 2\pi\left(-\frac{64}{3} + 16 + \frac{1}{3} - 1\right) = 2\pi(-21 + 16 - 1) = -12\pi$

(continued)

Top Disk T:

g. Parametrize the top disk T. (Start from cylindrical coordinates.)

Solution: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 3)$

h. Compute the normal vector and check its orientation:

Solution:
$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r & = & (\ \cos\theta, & \sin\theta, & 0) \\ \vec{e}_\theta & = & (-r\sin\theta, & r\cos\theta, & 0) \end{array} \qquad \vec{N} = \langle 0,0,r\rangle$$
This points up as required.

i. Evaluate $\vec{F} = \langle x,y,2z\rangle$ on the top disk:

Solution: $\vec{F}\Big|_{\vec{R}(r,\theta)} = \langle r\cos\theta, r\sin\theta, 6\rangle$

j. Compute the flux through T: (Be careful with the bounds on r.)

Solution: $\iint_T \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^4 \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^4 6r\,dr\,d\theta = 2\pi\left[3r^2\right]_0^4 = 96\pi$

Bottom Disk B:

k. Parametrize the bottom disk B. (Start from cylindrical coordinates.)

Solution: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$

l. Compute the normal vector and check its orientation:

Solution:
$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r & = & (\ \cos\theta, & \sin\theta, & 0) \\ \vec{e}_\theta & = & (-r\sin\theta, & r\cos\theta, & 0) \end{array} \qquad \vec{N} = \langle 0,0,r\rangle$$
Backwards. Reverse: $\vec{N} = \langle 0,0,-r\rangle$

m. Evaluate $\vec{F} = \langle x,y,2z\rangle$ on the bottom disk:

Solution: $\vec{F}\Big|_{\vec{R}(r,\theta)} = \langle r\cos\theta, r\sin\theta, 0\rangle$

n. Compute the flux through B: (Be careful with the bounds on r.)

Solution: $\iint_B \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \vec{F}\cdot\vec{N}\,dr\,d\theta = 0$

---

o. Compute the **TOTAL** right hand side:

Solution: $\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_T \vec{F}\cdot d\vec{S} + \iint_B \vec{F}\cdot d\vec{S} + \iint_S \vec{F}\cdot d\vec{S} = 96\pi + 0 - 12\pi = 84\pi$

which agrees with (b).