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MATH 221 Exam 1 Version H Fall 2019

Section 204 Solutions P. Yasskin

1-9	/54	11	/15
10	/36	Total	/105

Multiple Choice: (6 points each. No part credit.)

1. Find the angle between the vectors $\vec{a} = \langle 1,2,1 \rangle$ and $\vec{b} = \langle 0,1,1 \rangle$.

a. $90°$

b. $60°$

c. $45°$

d. $30°$ Correct Choice

e. $0°$

Solution: $\vec{a} \cdot \vec{b} = 2 + 1 = 3$ $\quad |\vec{a}| = \sqrt{1+4+1} = \sqrt{6}$ $\quad \left|\vec{b}\right| = \sqrt{1+1} = \sqrt{2}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\left|\vec{b}\right|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2} \qquad \theta = 30°$$

2. Two tugboats are pulling on a barge with the forces: $\vec{F}_1 = \langle 4,2 \rangle$ and $\vec{F}_2 = \langle -2,1 \rangle$
They move the barge from $P = (1,0)$ to $Q = (2,4)$. Find the work done.

a. 12

b. 14 Correct Choice

c. 16

d. 18

e. 20

Solution: The force is $\vec{F} = \vec{F}_1 + \vec{F}_2 = \langle 4,2 \rangle + \langle -2,1 \rangle = \langle 2,3 \rangle$. The displacement is $\vec{D} = \overrightarrow{PQ} = Q - P = (2,4) - (1,0) = (1,4)$. So the work is $W = \vec{F} \cdot \vec{D} = 2 + 12 = 14$.

3. Is the permutation $p = (3,5,2,6,1,4)$ odd or even? Find its inverse permutation.

a. Odd $\bar{p} = (5,3,1,6,2,4)$

b. Odd $\bar{p} = (4,2,6,1,3,5)$

c. Even $\bar{p} = (5,3,1,6,2,4)$ Correct Choice

d. Even $\bar{p} = (4,2,6,1,3,5)$

Solution: A sequence of transpositions is:
$(3,5,2,6,1,4) \to (1,5,2,6,3,4) \to (1,2,5,6,3,4) \to (1,2,3,6,5,4) \to (1,2,3,4,5,6)$
Since there are 4 transpositions, p is even.
Put p below $(1,2,3,4,5,6)$ and sort it carrying the top row:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 6 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 5 & 3 & 1 & 6 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$ So $\bar{p} = (5,3,1,6,2,4)$.

4. Compute the determinant: $D = \begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 3 & 4 & 0 \\ 0 & 4 & 5 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix}$

a. -11 Correct Choice

b. 11

c. -13

d. 13

e. 0

Solution: Expand on the 1^{st} row, then on the new 3^{rd} row.

$$D = 2\begin{vmatrix} 3 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 6 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 & 4 \\ 0 & 4 & 5 \\ 1 & 0 & 0 \end{vmatrix} = 2 \cdot 6\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 1 \cdot 1\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = (12 - 1)(15 - 16) = -11$$

5. Find the area of the triangle with vertices $A = (2,3,4)$, $B = (4,3,2)$ and $C = (4,2,4)$.

a. $\sqrt{3}$

b. $\sqrt{6}$ Correct Choice

c. 6

d. $\sqrt{12}$

e. 12

Solution: Two edges are $\overrightarrow{AB} = B - A = \langle 2,0,-2\rangle$ and $\overrightarrow{AC} = C - A = \langle 2,-1,0\rangle$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & -2 \\ 2 & -1 & 0 \end{vmatrix} = \hat{\imath}(0 - 2) - \hat{\jmath}(0 - -4) + \hat{k}(-2 - 0) = \langle -2,-4,-2\rangle$$

$$A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{4 + 16 + 4} = \frac{1}{2}\sqrt{24} = \sqrt{6}$$

6. Find a vector $\vec{w}$ of length 6 in the same direction as $\vec{v} = \langle 2,1,-2\rangle$. The sum of its components is

a. 12

b. 8

c. 6

d. 2 Correct Choice

e. 1

Solution: $|\vec{v}| = \sqrt{4 + 1 + 4} = 3$ $\quad \hat{v} = \dfrac{\vec{v}}{|\vec{v}|} = \left\langle \dfrac{2}{3}, \dfrac{1}{3}, \dfrac{-2}{3} \right\rangle$

We want $|\vec{w}| = 6$ and $\hat{w} = \hat{v} = \left\langle \dfrac{2}{3}, \dfrac{1}{3}, \dfrac{-2}{3} \right\rangle$. So $\vec{w} = |\vec{w}|\hat{w} = 6\left\langle \dfrac{2}{3}, \dfrac{1}{3}, \dfrac{-2}{3} \right\rangle = \langle 4,2,-4\rangle$.

The sum of its components is $4 + 2 - 4 = 2$.

7. Classify the surface: $2x^2 - 8x - y^2 + 6y + z^2 = 2$.

a. Hyperboloid of 1 sheet Correct Choice

b. Hyperboloid of 2 sheets

c. Cone

d. Hyperbolic Paraboloid

e. Hyperbolic Cylinder

Solution: Complete the squares: $2(x^2 - 2x + 4) - (y^2 - 6x + 9) + z^2 = 2 + 8 - 9$
$2(x-2)^2 - (y-3)^2 + z^2 = 1$ Hyperboloid of 1 sheet

8. Find the point where the line $(x,y,z) = (1+3t, 2+2t, 3+t)$ intersects the plane $2x - y + z = 13$. The sum of the components is:

a. -6

b. 6

c. 12

d. 18 Correct Choice

e. No intersection. They are parallel.

Solution: Substitute the line into the plane and solve for t:
$2(1+3t) - (2+2t) + 3 + t = 13$ or $3 + 5t = 13$ or $t = 2$
Substitute back into the line: $(x,y,z) = (1 + 3\cdot 2, 2 + 2\cdot 2, 3 + 2) = (7,6,5)$
Check in the plane: $2 \cdot 7 - 6 + 5 = 13$
The sum of the components is: $7 + 6 + 5 = 18$

9. Find the plane through the point $P = (0,5,3)$ with tangent vectors $\vec{u} = \langle 2,1,3\rangle$ and $\vec{v} = \langle -1,2,-2\rangle$. Its z-intercept is:

a. $z = 2$

b. $z = 4$ Correct Choice

c. $z = 5$

d. $z = 10$

e. $z = 20$

Solution: The normal is

$$\vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 3 \\ -1 & 2 & -2 \end{vmatrix} = \hat{\imath}(-2-6) - \hat{\jmath}(-4+3) + \hat{k}(4+1) = \langle -8,1,5\rangle$$

The plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $-8x + y + 5z = -8(0) + (5) + 5(3) = 20$.
The z-intercept satisfies $5z = 20$ or $z = 4$.

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (36 points) For the curve $\vec{r}(t) = \langle t, 2e^t, e^{2t} \rangle$ compute each of the following:

a. (6 pts) The velocity $\vec{v}$

Solution: $\vec{v} = \underline{\langle 1, 2e^t, 2e^{2t} \rangle}$

b. (6 pts) The speed $\dfrac{ds}{dt}$ (Simplify!)

Solution: $\dfrac{ds}{dt} = |\vec{v}| = \sqrt{1 + 4e^{2t} + 4e^{4t}} = \sqrt{(1 + 2e^{2t})^2} = 1 + 2e^{2t}$

$\dfrac{ds}{dt} = \underline{1 + 2e^{2t}}$

c. (6 pts) The tangential acceleration a_T

Solution: $a_T = \dfrac{d|\vec{v}|}{dt} = \dfrac{d}{dt}(1 + 2e^{2t}) = 4e^{2t}$

$a_T = \underline{4e^{2t}}$

d. (6 pts) The length of this curve between $(0,2,1)$ and $(1,2e,e^2)$.

Solution: $|\vec{v}| = 1 + 2e^{2t}$ $\quad (0,2,1) = \vec{r}(0) \quad (1,2e,e^2) = \vec{r}(1)$

$$L = \int_{(0,2,1)}^{(1,2e,e^2)} ds = \int_0^1 |\vec{v}|\,dt = \int_0^1 (1 + 2e^{2t})\,dt = \Big[t + e^{2t} \Big]_0^1 = 1 + e^2 - 1 = e^2$$

$L = \underline{e^2}$

e. (6 pts) The average value of the temperature along this curve between $(0,2,1)$ and $(1,2e,e^2)$ if the temperature is $T = yz$.

Solution: $|\vec{v}| = 1 + 2e^{2t}$ $\quad T = yz = 2e^t e^{2t} = 2e^{3t}$ $\quad T_{\text{ave}} = \dfrac{1}{L} \displaystyle\int_{(0,2,1)}^{(1,2e,e^2)} T\,ds$

$$\int_{(0,2,1)}^{(1,2e,e^2)} T\,ds = \int_0^1 yz|\vec{v}|\,dt = \int_0^1 2e^{3t}(1 + 2e^{2t})\,dt = \left[\frac{2e^{3t}}{3} + \frac{4e^{5t}}{5} \right]_0^1 = \frac{2e^3}{3} + \frac{4e^5}{5} - \frac{2}{3} - \frac{4}{5}$$

$$T_{\text{ave}} = \frac{1}{e^2}\left(\frac{2e^3}{3} + \frac{4e^5}{5} - \frac{2}{3} - \frac{4}{5} \right) = \frac{2e}{3} + \frac{4e^3}{5} - \frac{2}{3e^2} - \frac{4}{5e^2}$$

$T_{\text{ave}} = \underline{\dfrac{2e}{3} + \dfrac{4e^3}{5} - \dfrac{2}{3e^2} - \dfrac{4}{5e^2}}$

f. (6 pts) The work done to move a bead along of a wire in the shape of this curve between $(0,2,1)$ and $(1,2e,e^2)$ by the force $\vec{F} = \langle 0, z, y \rangle$.

Solution: $\vec{F}(\vec{r}(t)) = \langle 0, z, y \rangle = \langle 0, e^{2t}, 2e^t \rangle$ $\quad \vec{v} = \langle 1, 2e^t, 2e^{2t} \rangle$

$\vec{F} \cdot \vec{v} = e^{2t} \cdot 2e^t + 2e^t \cdot 2e^{2t} = 6e^{3t}$

$$W = \int_{(0,2,1)}^{(1,2e,e^2)} \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F} \cdot \vec{v}\,dt = \int_0^1 6e^{3t}\,dt = \Big[2e^{3t} \Big]_0^1 = 2e^3 - 2$$

$W = \underline{2e^3 - 2}$

11. (15 points) Consider the two straight lines:

$$L_1 : \quad (x,y,z) = (2+t, 3, 4+2t)$$
$$L_2 : \quad (x,y,z) = (1, 2+t, 3-2t)$$

Are they parallel or skew or do they intersect? If they intersect, find the point of intersection

Solution: The direction vectors are $\vec{v}_1 = \langle 1,0,2 \rangle$ and $\vec{v}_2 = \langle 0,1,-2 \rangle$. Since one is not a multiple of the other, the lines are not parallel. We first change the parameter name on the second line:

$$L_2 : \quad (x,y,z) = (1, 2+s, 3-2s)$$

We equate the x, y and z components:

$$2+t = 1$$
$$3 = 2+s$$
$$4+2t = 3-2s$$

The first two equations say $t = -1$ and $s = 1$. Using these, the third equation says

$$4+2(-1) = 3-2(1) \quad \text{or} \quad 2 = 1$$

This is impossible. So there is no solution. There is no intersection. The lines are skew!