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MATH 221 Exam 2 Version H Fall 2019

Section 204 Solutions P. Yasskin

1-8	/48	11	/16
9	/10	12	/25
10	/ 5	Total	/104

Multiple Choice: (6 points each. No part credit.)

1. Find the equation of the plane tangent to $z = x^3y + xy^2$ at the point $(x,y) = (1,2)$.
Its z-intercept is:

a. $c = -14$ Correct Choice

b. $c = -12$

c. $c = -6$

d. $c = 6$

e. $c = 14$

Solution: $f(x,y) = x^3y + xy^2$ $f_x(x,y) = 3x^2y + y^2$ $f_y(x,y) = x^3 + 2xy$

$f(1,2) = 1^3 \cdot 2 + 1 \cdot 2^2 = 6$ $f_x(1,2) = 3 \cdot 1^2 \cdot 2 + 2^2 = 10$ $f_y(1,2) = 1^3 + 2 \cdot 1 \cdot 2 = 5$

Tangent plane: $z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 6 + 10(x-1) + 5(y-2)$

$z = 10x + 5y - 14$ z-intercept is $c = -14$.

2. The volume of a frustum of a cone is $V = \frac{\pi}{3}(R^2 + Rr + r^2)h$ where R is the bottom radius, r is the top radius and h is the height. Currently, $R = 2$ cm, $r = 1$ cm and $h = 3$ cm.
Use differentials to estimate the change in volume if R and r increase by 0.1 cm while h decreases by 0.3.

a. $\Delta V \approx 3.2\pi$

b. $\Delta V \approx 1.6\pi$

c. $\Delta V \approx 0.8\pi$

d. $\Delta V \approx 0.6\pi$

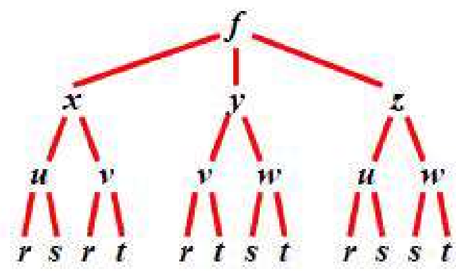
e. $\Delta V \approx 0.2\pi$ Correct Choice

Solution: $\Delta V \approx dV = \frac{\partial V}{\partial R}dR + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{\pi}{3}(2R + r)h\,dR + \frac{\pi}{3}(R + 2r)h\,dr + \frac{\pi}{3}(R^2 + Rr + r^2)\,dh$

$R = 2$ $r = 1$ $h = 3$ $dR = 0.1$ $dr = 0.1$ $dh = -0.3$

$\Delta V \approx \frac{\pi}{3}(2 \cdot 2 + 1)3 \cdot 0.1 + \frac{\pi}{3}(2 + 2 \cdot 1)3 \cdot 0.1 - \frac{\pi}{3}(2^2 + 2 \cdot 1 + 1^2)0.3 = \pi(.5 + .4 - .7) = 0.2\pi$

3. At the right is a tree diagram showing f as a function of x, y and z which are functions of u, v and w which are functions of r, s and t as indicated. Below are values of a bunch partial derivatives. Use this information to compute $\frac{\partial f}{\partial r}$.



$$\frac{\partial f}{\partial x} = 2 \qquad \frac{\partial f}{\partial y} = 3 \qquad \frac{\partial f}{\partial z} = 4$$

$$\frac{\partial x}{\partial u} = 5 \qquad \frac{\partial x}{\partial v} = 6 \qquad \frac{\partial y}{\partial v} = 7 \qquad \frac{\partial y}{\partial w} = 8 \qquad \frac{\partial z}{\partial u} = 9 \qquad \frac{\partial z}{\partial w} = 10$$

$$\frac{\partial u}{\partial r} = 6 \qquad \frac{\partial u}{\partial s} = 5 \qquad \frac{\partial v}{\partial r} = 4 \qquad \frac{\partial v}{\partial t} = 3 \qquad \frac{\partial w}{\partial s} = 2 \qquad \frac{\partial w}{\partial t} = 1$$

a. 163

b. 212

c. 358

d. 396

e. 408 Correct Choice

Solution
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\frac{\partial u}{\partial r}$$
$$= 2 \cdot 5 \cdot 6 + 2 \cdot 6 \cdot 4 + 3 \cdot 7 \cdot 4 + 4 \cdot 9 \cdot 6 = 408$$

4. The point $(x,y) = (-1,2)$ is a critical point of the function $f = 8x^3 - y^3 - 12xy$. Use the 2^{nd} Derivative Test to classify it as:

a. Local Minimum

b. Local Maximum Correct Choice

c. Inflection Point

d. Saddle Point

e. The 2^{nd} Derivative Test FAILS.

Solution:
$f_x = 24x^2 - 12y \qquad f_y = -3y^2 - 12x \qquad f_x(-1,2) = 24 - 12 \cdot 2 = 0 \qquad f_y(-1,2) = -3 \cdot 4 - 12(-1) = 0$

$f_{xx} = 48x \qquad f_{xx}(-1,2) = -48 \qquad f_{yy} = -6y \qquad f_{yy}(-1,2) = -12 \qquad f_{xy} = -12$

$D = f_{xx}f_{yy} - f_{xy}^2 = 48 \cdot 12 - 12^2 = 432$

$D > 0$ and $f_{xx} < 0$. So this is a local maximum.

5. If x, y and z are related by $x\cos y + z\sin y = 3$. Find $\frac{\partial z}{\partial x}$ at the point $(x,y,z) = \left(\sqrt{3}\,, \frac{\pi}{6}, 3\right)$.

a. $\frac{1}{\sqrt{3}}$

b. $\frac{-1}{\sqrt{3}}$

c. $\sqrt{3}$

d. $-\sqrt{3}$ Correct Choice

e. $\frac{1}{3}$

Solution: Apply $\frac{\partial}{\partial x}$: $\cos y + \frac{\partial z}{\partial x}\sin y = 0$ $\frac{\sqrt{3}}{2} + \frac{\partial z}{\partial x}\frac{1}{2} = 0$ $\frac{\partial z}{\partial x} = -\sqrt{3}$

6. If x, y and z are related by $x\cos y + z\sin y = 3$. Find $\frac{\partial z}{\partial t}$ at the instant when:

$$(x,y,z) = \left(\sqrt{3}\,, \frac{\pi}{6}, 3\right) \qquad \frac{dx}{dt} = \frac{1}{\sqrt{3}} \qquad \frac{dy}{dt} = \frac{1}{\sqrt{3}}$$

a. -1

b. -2

c. -3 Correct Choice

d. $-\sqrt{3}$

e. $\frac{-1}{\sqrt{3}}$

Solution: Apply $\frac{d}{dt}$: $\frac{dx}{dt}\cos y - x\sin y\frac{dy}{dt} + \frac{\partial z}{\partial t}\sin y + z\cos y\frac{dy}{dt} = 0$

Plug in numbers: $\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} - \sqrt{3}\,\frac{1}{2}\frac{1}{\sqrt{3}} + \frac{\partial z}{\partial t}\frac{1}{2} + 3\frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}} = 0$

Multiply by 2: $1 - 1 + \frac{\partial z}{\partial t} + 3 = 0$ $\frac{\partial z}{\partial t} = -3$

7. Find the tangent plane to the graph of the equation $xy - zy = -4$ at the point $(x,y,z) = (1,2,3)$. Its z-intercept is:

a. $c = -8$

b. $c = -4$

c. $c = 0$

d. $c = 4$ Correct Choice

e. $c = 8$

Solution: Let $f = xy - zy$ and $P = (1,2,3)$. Then $\vec{\nabla} f = \langle y, x - z, -y\rangle$

$\vec{N} = \vec{\nabla} f\big|_P = \langle 2,-2,-2\rangle$ $\vec{N}\cdot X = \vec{N}\cdot P$ $2x - 2y - 2z = 2\cdot 1 - 2\cdot 2 - 2\cdot 3 = -8$

The z-intercept is $c = \frac{-8}{-2} = 4$.

8. Queen Lena is flying the Centurion Eagle through a deadly Sythion field whose density is $S = xyz \dfrac{\text{Sythions}}{\text{microlightyear}^3}$. The top speed of the Centurion Eagle is $14 \dfrac{\text{microlightyears}}{\text{lightyear}}$. If Lena is located at the point $(x,y,z) = (3,2,1)$, what should her velocity be to **decrease** the Sythion density as fast as possible?

a. $\langle -4,-6,-12\rangle$ Correct Choice

b. $\langle -2,-3,-6\rangle$

c. $\langle -28,42,-84\rangle$

d. $\langle 4,6,12\rangle$

e. $\langle 2,3,6\rangle$

Solution:

$\vec{\nabla} S = \langle yz,xz,xy\rangle \qquad \vec{\nabla} S\Big|_{(3,2,1)} = \langle 2,3,6\rangle \qquad \left|\vec{\nabla} S\right| = \sqrt{4+9+36} = 7 \qquad \widehat{\nabla S} = \dfrac{\vec{\nabla} S}{\left|\vec{\nabla} S\right|} = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\rangle$

The direction of maximum decrease is $\vec{v} = -\widehat{\nabla S} = \left\langle \frac{-2}{7}, \frac{-3}{7}, \frac{-6}{7}\right\rangle$. The maximum speed is $|\vec{v}| = 14$.

So the velocity of maximum decrease is $\vec{v} = |\vec{v}|\hat{v} = 14\left\langle \frac{-2}{7}, \frac{-3}{7}, \frac{-6}{7}\right\rangle = \langle -4,-6,-12\rangle$.

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (10 points) Prove whether the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{x^4+2x^2y^2+y^4}$ converges or diverges. If it converges, find the limit.

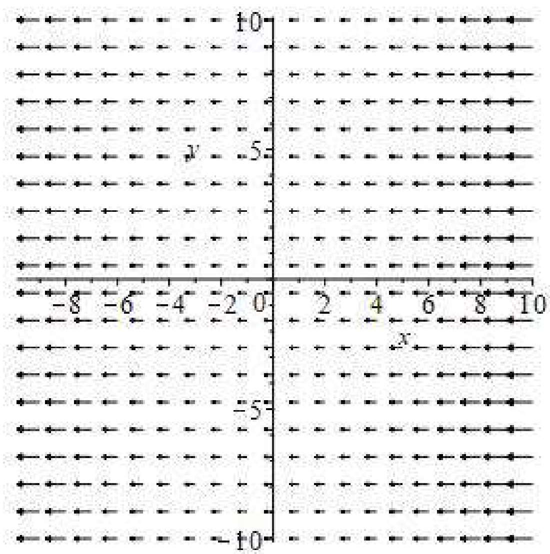
Solution: First notice $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{x^4+2x^2y^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{(x^2+y^2)^2}$.

Switch to polar coordinates:

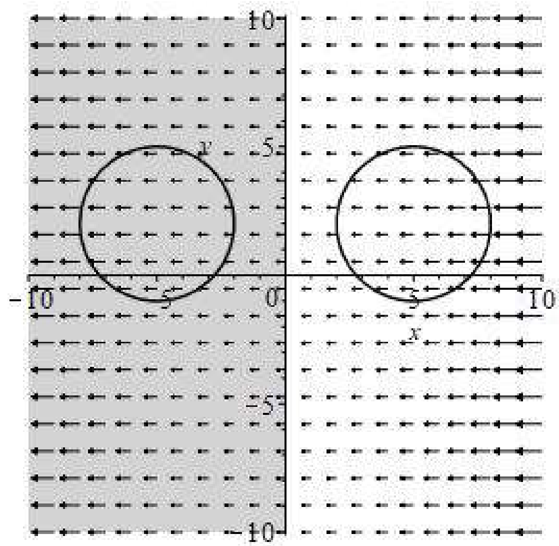
$$\lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{x^4+2x^2y^2+y^4} = \lim_{r\to 0} \frac{r^6\cos^6\theta + r^6\sin^6\theta}{r^4} = \lim_{r\to 0} r^2(\cos^6\theta + \sin^6\theta) = 0$$

because $r^2 \to 0$ and $0 \le \cos^6\theta + \sin^6\theta \le 2$.

10. (5 points) Here is the plot of a vector field $\vec{F}$ in $\mathbb{R}^2$.
Shade in the region where $\vec{\nabla} \cdot \vec{F} > 0$. Explain why.



Solution: On the left half of the plot larger arrows point out than in on each circle. So on the left, $\vec{\nabla} \cdot \vec{F} > 0$.



11. (16 points) Let $\vec{F} = \langle x, 2y, -3z \rangle$.

a. Find a scalar potential, f, for $\vec{F}$ or show one does not exist.
Explain all steps neatly and clearly.

Solution: By inspection $f = \frac{1}{2}x^2 + y^2 - \frac{3}{2}z^2$

b. Find a vector potential, $\vec{A}$, for $\vec{F}$ or show one does not exist.
Explain all steps neatly and clearly.

Solution: $\vec{\nabla} \times \vec{A} = \vec{F}$ We look for a scalar potential with $A_3 = 0$. The equations are:

(1) $-\partial_z A_2 = x$ (2) $\partial_z A_1 = 2y$ (3) $\partial_x A_2 - \partial_y A_1 = -3z$

(1) $\Rightarrow$ $\partial_z A_2 = -x$ $A_2 = -xz + f(x,y)$

(2) $\Rightarrow$ $\partial_z A_1 = 2y$ $A_1 = 2yz + g(x,y)$

(3) $\Rightarrow$ $\partial_x A_2 - \partial_y A_1 = \partial_x(-xz + f(x,y)) - \partial_y(2yz + g(x,y)) = -z + \partial_x f - 2z - \partial_y g = -3z$

Take $f = g = 0$.

$$\vec{A} = \langle 2yz, -xz, 0 \rangle$$

Check: $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2yz & -xz & 0 \end{vmatrix} = \hat{\imath}(0 - -x) - \hat{\jmath}(0 - 2y) + \hat{k}(-z - 2z) = \langle x, 2y, -3z \rangle = \vec{F}$

12. (25 points) Find the largest and smallest values of the function $f(x,y,z) = xyz$ on the ellipsoid $x^2 + 4y^2 + 9z^2 = 108$.

Solution Method 1: Lagrange Multipliers:
Let $g = x^2 + 4y^2 + 9z^2$.

$$\vec{\nabla} f = \langle yz, xz, xy \rangle \qquad\qquad \vec{\nabla} g = \langle 2x, 8y, 18z \rangle$$

Lagrange equations: $\vec{\nabla} f = \lambda \vec{\nabla} g$

$$yz = \lambda 2x \qquad xz = \lambda 8y \qquad xy = \lambda 18z$$

Multiply first equation by x. Second by y. Third by z.

$$xyz = \lambda 2x^2 \qquad xyz = \lambda 8y^2 \qquad xyz = \lambda 18z^2$$

Equate and divide by 2λ:

$$x^2 = 4y^2 = 9z^2$$

Substitute into the elipsoid:

$$x^2 + x^2 + x^2 = 108 \qquad x^2 = 36 \qquad x = \pm 6 \qquad y = \pm 3 \qquad z = \pm 2$$

The function values are

$$f = xyz = (\pm 6)(\pm 3)(\pm 2) = \pm 36$$

Maximum is $f = 36$. Minimum is $f = -36$.

Solution Method 2: Eliminate a Variable:
It is easier to extremize $F = f^2 = x^2y^2z^2$ and then take a square root.
We solve the constraint for $x^2 = 108 - 4y^2 - 9z^2$ and plug into F:

$$F = (108 - 4y^2 - 9z^2)y^2z^2 = 108y^2z^2 - 4y^4z^2 - 9y^2z^4$$

$$F_y = 216yz^2 - 16y^3z^2 - 18yz^4 = 2yz^2(108 - 8y^2 - 9z^2) = 0$$

$$F_z = 216y^2z - 8y^4z - 36y^2z^3 = 4y^2z(54 - 2y^2 - 9z^2) = 0$$

Case 1: $y = 0$ Then $f = 0$. Note: x and z are anything satisfying $x^2 + 9z^2 = 108$.
Case 2: $z = 0$ Then $f = 0$. Note: x and y are anything satisfying $x^2 + 4y^2 = 108$.
Case 3: $y \neq 0$ and $z \neq 0$ Then

$$8y^2 + 9z^2 = 108 \qquad (1)$$

$$2y^2 + 9z^2 = 54 \qquad (2)$$

(1) - (2) says: $6y^2 = 54$ or $y = \pm 3$.
Then (2) says: $9z^2 = 54 - 18 = 36$ or $z = \pm 2$.
Then the constraint says: $x^2 = 108 - 4y^2 - 9z^2 = 108 - 36 - 36 = 36$ or $x = \pm 6$
The function values are

$$f = xyz = (\pm 6)(\pm 3)(\pm 2) = \pm 36$$

Maximum is $f = 36$. Minimum is $f = -36$.
The $f = 0$ values from Cases 1 and 2 don't matter.