Name____________________ ID________________		
	1-12	/60
	13	/20
	14	/20
	15	/10
	Total	/110

MATH 251 Final Exam Fall 2005

Sections 503 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the area of the triangle with vertices $(1,1,1)$, $(2,2,4)$ and $(1,3,9)$.

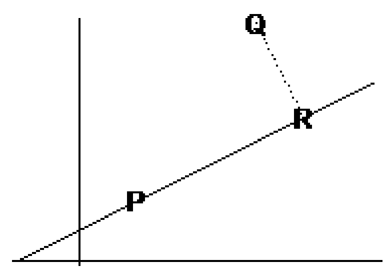
- **a.** $\sqrt{2}$
- **b.** $3\sqrt{2}$ Correct Choice
- **c.** $6\sqrt{2}$
- **d.** 4
- **e.** 2

$\vec{u} = (2,2,4) - (1,1,1) = (1,1,3) \qquad \vec{v} = (1,3,9) - (1,1,1) = (0,2,8)$

$$\vec{u} \times \vec{v} = (1,1,3) \times (0,2,8) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 3 \\ 0 & 2 & 8 \end{vmatrix} = \hat{\imath}(8-6) - \hat{\jmath}(8) + \hat{k}(2) = (2,-8,2)$$

$$A = \frac{1}{2}|\vec{u} \times \vec{v}| = \frac{1}{2}\sqrt{4+64+4} = \frac{1}{2}\sqrt{72} = 3\sqrt{2}$$

2. If you drop a perpendicular from the point $Q = (3,4)$ to the line through $P = (1,1)$ with direction $\vec{v} = (2,1)$, then the foot of the perpendicular is $R =$



- **a.** $(3.2, 2.0)$
- **b.** $(3.4, 2.2)$
- **c.** $(3.6, 2.3)$
- **d.** $(3.8, 2.4)$ Correct Choice
- **e.** $(4.0, 2.5)$

$$\overrightarrow{PQ} = (2,3) \qquad proj_{\vec{v}}\overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \frac{7}{5}(2,1) = \left(\frac{14}{5}, \frac{7}{5}\right)$$

$$R = P + proj_{\vec{v}}\overrightarrow{PQ} = (1,1) + \left(\frac{14}{5}, \frac{7}{5}\right) = \left(\frac{19}{5}, \frac{12}{5}\right) = (3.8, 2.4)$$

3. Find the point where the line $(x,y,z) = (3,2,1) + t(1,2,3)$ intersects the plane $x - y + z = -2$.
At this point, $x + y + z =$

- **a.** -6 Correct Choice
- **b.** -2
- **c.** 0
- **d.** 2
- **e.** 6

$-2 = x - y + z = (3 + t) - (2 + 2t) + (1 + 3t) = 2t + 2 \qquad t = -2$
$(x,y,z) = (3,2,1) - 2(1,2,3) = (1,-2,-5) \qquad 1 - 2 - 5 = -6$

4. Use the linear approximation to $\ln(x) + 2\ln(y)$ at $(x,y) = (1,1)$ to estimate $\ln(1.3) + 2\ln(.9)$.

- **a.** 1.1
- **b.** 1.05
- **c.** 0.5
- **d.** 0.1 Correct Choice
- **e.** 0.05

$f(x,y) = \ln(x) + 2\ln(y) \qquad f_x(x,y) = \frac{1}{x} \qquad f_y(x,y) = \frac{2}{y}$
$f(1,1) = 0 \qquad f_x(1,1) = 1 \qquad f_y(1,1) = 2$
$f_{\tan}(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$
$= 0 + 1(1.3 - 1) + 2(.9 - 1) = (.3) + 2(-.1) = 0.1$

5. Find the equation of the plane tangent to the hyperboloid $z^2 - x^2 - y^2 = 4$ at the point $(1,2,3)$.

- **a.** $-2x + 4y + 6z = 24$
- **b.** $x + 2y + 3z = 14$
- **c.** $2x + 4y - 6z = -8$ Correct Choice
- **d.** $(x,y,z) = (1 - 2t, 2 - 4t, 3 + 6t)$
- **e.** $(x,y,z) = (1 - 2t, 2 + 4t, 3 + 6t)$

$P = (1,2,3) \qquad F = z^2 - x^2 - y^2 \qquad \vec{\nabla}F = (-2x,-2y,2z) \qquad \vec{N} = \vec{\nabla}F\Big|_{(1,2,3)} = (-2,-4,6)$
$\vec{N} \cdot X = \vec{N} \cdot P \qquad -2x - 4y + 6z = -2 - 8 + 18 = 8$

6. Duke Skywater is travelling through the galaxy. He is currently at the point with galactic coordinates $(40,25,53)$ (in lightyears), and his velocity is $(.2,-.1,.3)$ (in lightyears/year). He measures the polaron density to be $U = 4300$ polarons/cm^3 and its gradient to be $\vec{\nabla}U = (3,2,1)$ polarons/cm^3/lightyear. Find the rate at which he sees the polaron density changing.

a. 223

b. 21.4

c. 0.11

d. -0.11

e. 0.7 Correct Choice

$$\frac{dU(\vec{r}(t))}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} = \vec{\nabla}U \cdot \vec{v} = (3,2,1)\cdot(.2,-.1,.3) = .6 - .2 + .3 = .7$$

7. The function $f(x,y) = \frac{1}{4}x^3 - xy + \frac{2}{27}y^3$ has a critical point at $(x,y) = (2,3)$ which, according to the Second Derivative Test, is

a. a local minimum. Correct Choice

b. a local maximum.

c. a saddle point.

d. an inflection point.

e. The Second Derivative Test FAILS.

$$f_x = \frac{3}{4}x^2 - y \qquad f_y = -x + \frac{2}{9}y^2$$

$$f_{xx} = \frac{3}{2}x \qquad f_{yy} = \frac{4}{9}y \qquad f_{xy} = -1 \qquad D = f_{xx}f_{yy} - f_{xy}{}^2 = \frac{2}{3}xy - 1$$

$$f_{xx}(2,3) = \frac{3}{2}2 = 3 > 0 \qquad D(2,3) = \frac{2}{3}2\cdot 3 - 1 = 3 > 0$$ local minimum

8. Rewrite the polar equation $r^2 = \sin 2\theta$ in rectangular coordinates.

a. $x^4 + y^4 = 2xy$

b. $(x^2+y^2)^2 = 2xy$ Correct Choice

c. $(x^2+y^2)^{3/2} = 2y$

d. $(x^2+y^2)^{3/2} = 2x$

e. $x^3 + y^3 = 2y$

$$r^2 = \sin 2\theta = 2\sin\theta\cos\theta = 2\frac{y}{r}\frac{x}{r} \quad\Rightarrow\quad r^4 = 2xy \quad\Rightarrow\quad (x^2+y^2)^2 = 2xy$$

9. For an ideal gas, the pressure, P, is a function of the temperature, T, and volume, V, given by $P = \frac{kT}{V}$ where k is a constant. For a certain sample of gas the current values are

$$T = 250°\text{K} \qquad V = 5\text{ m}^3 \qquad k = 2\,\frac{\text{kPa}\cdot\text{m}^3}{°\text{K}} \qquad \text{and consequently} \qquad P = 100\text{ kPa}$$

If the volume and temperature are increasing at

$$\frac{dV}{dt} = 0.2\,\frac{\text{m}^3}{\text{sec}} \qquad \text{and} \qquad \frac{dT}{dt} = 6\,\frac{°\text{K}}{\text{sec}}$$

is the pressure increasing or decreasing and at what rate?

a. decreasing at $1.6\,\frac{\text{kPa}}{\text{sec}}$ Correct Choice

b. decreasing at $8\,\frac{\text{kPa}}{\text{sec}}$

c. increasing at $1.6\,\frac{\text{kPa}}{\text{sec}}$

d. increasing at $8\,\frac{\text{kPa}}{\text{sec}}$

e. Pressure is constant.

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{k}{V}\frac{dT}{dt} - \frac{kT}{V^2}\frac{dV}{dt} = \frac{2\cdot 6}{5} - \frac{2\cdot 250\cdot 0.2}{25} = -1.6\,\frac{\text{kPa}}{\text{sec}}$$

10. A particle moves along the curve $\vec{r}(t) = (t, t^2, t^3)$ from $(1,1,1)$ to $(2,4,8)$ due to the force $\vec{F} = (z, y, x)$. Find the work done by the force.

a. $\frac{70}{3}$

b. 24

c. $\frac{45}{2}$ Correct Choice

d. $\frac{96}{5}$

e. $\frac{93}{5}$

$$\vec{v} = (1, 2t, 3t^2) \qquad \vec{F}(\vec{r}(t)) = (t^3, t^2, t) \qquad \vec{F}\cdot\vec{v} = t^3 + 2t^3 + 3t^3 = 6t^3$$

$$W = \int_{(1,1,1)}^{(2,4,8)} \vec{F}\cdot d\vec{s} = \int_1^2 \vec{F}\cdot\vec{v}\,dt = \int_1^2 6t^3\,dt = \left[\frac{3t^4}{2}\right]_1^2 = \frac{48}{2} - \frac{3}{2} = \frac{45}{2}$$

11. Compute $\int_{(1,1,1)}^{(1/e,\,1/e,\,e^2)} \vec{F} \cdot d\vec{s}$ where $\vec{F} = (z, z, x+y)$ along the curve $\vec{r}(t) = (e^{-t}, e^{-t}, e^{2t})$.

HINT: Note $\vec{F} = \vec{\nabla} f$ where $f = xz + yz$.

a. $4 - 4e$

b. $4e - 4$

c. $4 - 2e$

d. $2 - 2e$

e. $2e - 2$ Correct Choice

By the Fundamental Theorem of Calculus for Curves,

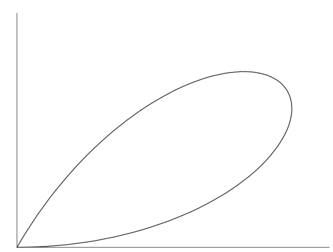
$$\int_{(1,1,1)}^{(1/e,1/e,e^2)} \vec{F} \cdot d\vec{s} = \int_{(1,1,1)}^{(1/e,1/e,e^2)} \vec{\nabla} f \cdot d\vec{s} = f\left(\frac{1}{e}, \frac{1}{e}, e^2\right) - f(1,1,1)$$
$$= \left(\frac{1}{e}e^2 + \frac{1}{e}e^2\right) - (1 \cdot 1 + 1 \cdot 1) = 2e - 2$$

12. Use Green's Theorem to compute

$$\oint (y^2 + 2y - \ln x)\, dx + (4x + 2xy + e^y)\, dy$$

counterclockwise around the polar curve

$$r = \sin(3\theta) \quad \text{for} \quad 0 \le \theta \le \frac{\pi}{3}.$$



a. $\frac{\pi}{2}$

b. $\frac{\pi}{3}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{6}$ Correct Choice

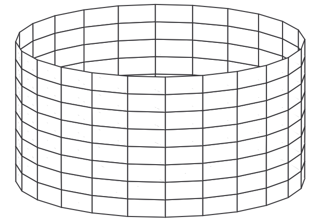
e. $\frac{\pi}{12}$

By Green's Theorem,

$$\oint (y^2 + 2y - \ln x)\, dx + (4x + 2xy + e^y)\, dy = \iint \left[\frac{\partial}{\partial x}(4x + 2xy + e^y) - \frac{\partial}{\partial y}(y^2 + 2y - \ln x) \right] dx\, dy$$
$$= \iint [(4 + 2y) - (2y + 2)]\, dx\, dy = \iint 2\, dx\, dy = \int_0^{\pi/3} \int_0^{\sin(3\theta)} 2\, r\, dr\, d\theta = \int_0^{\pi/3} \Big[r^2 \Big]_0^{\sin(3\theta)} d\theta$$
$$= \int_0^{\pi/3} \sin^2(3\theta)\, d\theta = \int_0^{\pi/3} \frac{1 - \cos(6\theta)}{2}\, d\theta = \left[\frac{1}{2}\theta \right]_0^{\pi/3} = \frac{\pi}{6}$$

Work Out: (Points indicated. Part credit possible.)

13. (20 points) Verify Stokes' Theorem $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$



for the vector field $\vec{F} = (-yz, xz, z^2)$ and

the cylinder $x^2 + y^2 = 4$ for $1 \le z \le 3$ oriented out.

Be sure to check and explain the orientations.

Use the following steps:

a. The cylindrical surface may be parametrized by $\vec{R}(\theta,z) = (2\cos\theta, 2\sin\theta, z)$.
Compute the surface integral:
Successively find: $\vec{e}_\theta$, $\vec{e}_z$, $\vec{N}$, $\vec{\nabla} \times \vec{F}$, $\vec{\nabla} \times \vec{F}\big(\vec{R}(\theta,z)\big)$, $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta & = & (-2\sin\theta, & 2\cos\theta, & 0) \\ \vec{e}_z & = & (\quad 0, & 0, & 1) \end{array}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(2\cos\theta) - \hat{\jmath}(-2\sin\theta) + \hat{k}(0) = (2\cos\theta, 2\sin\theta, 0)$$

$\vec{N}$ has the correct orientation.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz, & xz, & z^2 \end{vmatrix} = \hat{\imath}(0 - x) - \hat{\jmath}(0 - -y) + \hat{k}(z - -z) = (-x, -y, 2z)$$

$$\vec{\nabla} \times \vec{F}\big(\vec{R}(\theta,z)\big) = (-2\cos\theta, -2\sin\theta, 2z)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -4\cos^2\theta - 4\sin^2\theta = -4$$

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_C \vec{\nabla} \times \vec{F} \cdot \vec{N}\, d\theta\, dz = \int_1^3 \int_0^{2\pi} -4\, d\theta\, dz = 2\pi[-4z]_{z=1}^3 = -16\pi$$

b. Let U be the upper circle. Parametrize U and compute the line integral.

Successively find: $\vec{r}(\theta)$, $\vec{v}(\theta)$, $\vec{F}(\vec{r}(\theta))$, $\oint\limits_U \vec{F} \cdot d\vec{s}$.

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 3)$

$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$

By the right hand rule the upper curve must be traversed clockwise but $\vec{v}$ points counterclockwise. So reverse $\vec{v}$:

$\vec{v}(\theta) = (2\sin\theta, -2\cos\theta, 0)$

$\vec{F}(\vec{r}(\theta)) = (-yz, xz, z^2) = (-6\sin\theta, 6\cos\theta, 9)$

$$\oint\limits_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -12\sin^2\theta - 12\cos^2\theta\, d\theta = \int_0^{2\pi} -12\, d\theta = -24\pi$$

c. Let L be the lower circle. Parametrize L and compute the line integral.

Successively find: $\vec{r}(\theta)$, $\vec{v}(\theta)$, $\vec{F}(\vec{r}(\theta))$, $\oint\limits_L \vec{F} \cdot d\vec{s}$.

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 1)$

$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$

By the right hand rule the lower curve must be traversed counterclockwise and $\vec{v}$ is counterclockwise.

$\vec{F}(\vec{r}(\theta)) = (-yz, xz, z^2) = (-2\sin\theta, 2\cos\theta, 1)$

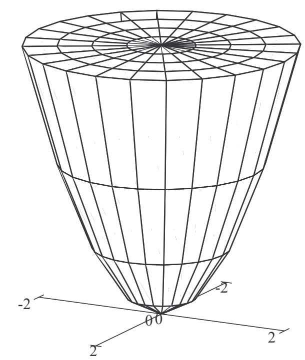
$$\oint\limits_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 4\sin^2\theta + 4\cos^2\theta\, d\theta = \int_0^{2\pi} 4\, d\theta = 8\pi$$

d. Combine $\oint\limits_U \vec{F} \cdot d\vec{s}$ and $\oint\limits_L \vec{F} \cdot d\vec{s}$ to get $\oint\limits_{\partial C} \vec{F} \cdot d\vec{s}$.

$$\oint\limits_{\partial C} \vec{F} \cdot d\vec{s} = \oint\limits_U \vec{F} \cdot d\vec{s} + \oint\limits_L \vec{F} \cdot d\vec{s} = -24\pi + 8\pi = -16\pi$$

which agrees with part (a).

14. (20 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$



for the vector field $\vec{F} = (xz, yz, z^2)$ and

the volume above the paraboloid P: $z = x^2 + y^2$ for $z \le 4$

and below the disk D: $x^2 + y^2 \le 4$ with $z = 4$.

Be sure to check and explain the orientations.

Use the following steps.

a. Compute the divergence $\vec{\nabla} \cdot \vec{F}$ and the volume integral $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV$.

$$\vec{\nabla} \cdot \vec{F} = z + z + 2z = 4z$$

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 4z\, r\, dz\, dr\, d\theta = 2\pi \int_0^2 \left[2z^2 \right]_{z=r^2}^4 r\, dr = 4\pi \int_0^2 (16 - r^4)\, r\, dr$$

$$= 4\pi \left[8r^2 - \frac{r^6}{6} \right]_0^2 = 4\pi \left(32 - \frac{32}{3} \right) = \frac{256}{3}\pi$$

b. Parametrize the disk, D, and compute the surface integral:

Successively find: $\vec{R}(r,\theta)$, $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$, $\vec{F}\big(\vec{R}(r,\theta)\big)$, $\iint_D \vec{F} \cdot d\vec{S}$.

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4)$$

	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\vec{e}_r =$	$(\cos\theta,$	$\sin\theta,$	$0)$
$\vec{e}_\theta =$	$(-r\sin\theta,$	$r\cos\theta,$	$0)$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (0, 0, r)$$

$\vec{N}$ has the correct orientation which is up.

$$\vec{F}\big(\vec{R}(r,\theta)\big) = (xz, yz, z^2) = (4r\cos\theta, 4r\sin\theta, 16)$$

$$\vec{F} \cdot \vec{N} = 16r$$

$$\iint_D \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^2 16r\, dr\, d\theta = 2\pi [8r^2]_0^2 = 64\pi$$

c. The paraboloid, P, may be parametrize as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.
Compute the surface integral:
Successively find: $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$, $\vec{F}\big(\vec{R}(r,\theta)\big)$, $\iint\limits_P \vec{F}\cdot d\vec{S}$.

$$\begin{array}{rccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 2r) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$\vec{N}$ needs to be oriented down. So reverse $\vec{N}$:

$$\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$$

$$\vec{F}\big(\vec{R}(r,\theta)\big) = (xz, yz, z^2) = (r^3\cos\theta, r^3\sin\theta, r^4)$$

$$\vec{F}\cdot\vec{N} = 2r^5\cos^2\theta + 2r^5\sin^2\theta - r^5 = r^5$$

$$\iint\limits_P \vec{F}\cdot d\vec{S} = \iint\limits_P \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^5\,dr\,d\theta = 2\pi\left[\frac{r^6}{6}\right]_0^2 = \frac{64}{3}\pi$$

d. Combine $\iint\limits_D \vec{F}\cdot d\vec{S}$ and $\iint\limits_P \vec{F}\cdot d\vec{S}$ to get $\iint\limits_{\partial V} \vec{F}\cdot d\vec{S}$.

$$\iint\limits_{\partial V} \vec{F}\cdot d\vec{S} = \iint\limits_D \vec{F}\cdot d\vec{S} + \iint\limits_P \vec{F}\cdot d\vec{S} = 64\pi + \frac{64}{3}\pi = \frac{256}{3}\pi$$

which agrees with part (a).

15. $\big($10 points$\big)$ Find the point on the plane $z = 2 - 2x - y$ that is closest to the origin.

Minimize $f = D^2 = x^2 + y^2 + z^2 = x^2 + y^2 + (2 - 2x - y)^2$

$$\left.\begin{array}{l} f_x = 2x - 4(2-2x-y) = 0 \\ f_y = 2y - 2(2-2x-y) = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} 10x + 4y = 8 \\ 4x + 4y = 4 \end{array}\right\} \Rightarrow \left.\begin{array}{c} 6x = 4 \\ x + y = 1 \end{array}\right\} \Rightarrow \begin{array}{l} x = \frac{2}{3} \\ y = \frac{1}{3} \end{array}$$