Name________________________ ID_______________

| 1-3 | /12 |
|---|---|
| 4 | / 8 |
| 5 | / 8 |
| Total | /28 |

MATH 251 Quiz 7 Fall 2005

Sections 503 Solutions P. Yasskin

Multiple Choice: (4 points each)

**1.** (4 points) If $\vec{F} = (x\sin y, z\cos y, x^2 + z^2)$, compute $\vec{\nabla} \cdot \vec{F}$.

**a.** $(\sin y, -z\sin y, 2z)$
**b.** $(-\cos y, -2x, -x\cos y)$
**c.** $(-\cos y, 2x, -x\cos y)$
**d.** $-\cos y - 2x - x\cos y$
**e.** $\sin y - z\sin y + 2z$ Correct Choice

$$\vec{\nabla} \cdot \vec{F} = \partial_x(x\sin y) + \partial_y(z\cos y) + \partial_z(x^2 + z^2) = \sin y - z\sin y + 2z$$

**2.** (4 points) If $\vec{F} = (x\sin y, z\cos y, x^2 + z^2)$, compute $\vec{\nabla} \times \vec{F}$.

**a.** $(\sin y, -z\sin y, 2z)$
**b.** $(-\cos y, -2x, -x\cos y)$ Correct Choice
**c.** $(-\cos y, 2x, -x\cos y)$
**d.** $-\cos y - 2x - x\cos y$
**e.** $\sin y - z\sin y + 2z$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x\sin y & z\cos y & x^2 + z^2 \end{vmatrix} \begin{array}{l} = \hat{\imath}(0 - \cos y) - \hat{\jmath}(2x - 0) + \hat{k}(0 - x\cos y) \\ = (-\cos y, -2x, -x\cos y) \end{array}$$

**3.** (4 points) If $\vec{F} = (x\sin y, z\cos y, x^2 + z^2)$, compute $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F}$.

**a.** $0$ Correct Choice
**b.** $-z\cos y + 2$
**c.** $z\cos y + 2$
**d.** $(x\sin y, -\cos y, -2 - \sin y)$
**e.** $(x\sin y, \cos y, -2 - \sin y)$

$\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ for any $\vec{F}$. In particular, $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = \partial_x(-\cos y) + \partial_y(-2x) + \partial_z(x - x\cos y) = 0$

**4.** (8 points) Compute $\iint \vec{F}\,d\vec{S}$ over the sphere $x^2+y^2+z^2=4$ with an outward normal for the vector field $\vec{F}=(3x,3y,6z)$.

Note: The sphere may be parametrized by $\vec{R}(\theta,\varphi)=(2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$.
Follow these steps:

$$\vec{e}_\theta = (-2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, \quad 0)$$

$$\vec{e}_\varphi = (2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi)$$

$$\vec{N} = \hat{\imath}(-4\sin^2\varphi\cos\theta) - \hat{\jmath}(4\sin^2\varphi\sin\theta) + \hat{k}(-4\sin\varphi\cos\varphi)$$

$$= (-4\sin^2\varphi\cos\theta, -4\sin^2\varphi\sin\theta, -4\sin\varphi\cos\varphi)$$

Reverse $\vec{N}$ to get it to point out:

$$\vec{N} = (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$$

$$\vec{F}\big(\vec{R}(\theta,\varphi)\big) = (6\sin\varphi\cos\theta, 6\sin\varphi\sin\theta, 12\cos\varphi)$$

$$\vec{F}\cdot\vec{N} = 24\sin^3\varphi\cos^2\theta + 24\sin^3\varphi\sin^2\theta + 48\sin\varphi\cos^2\varphi$$

$$= 24\sin^3\varphi + 48\sin\varphi\cos^2\varphi$$

$$= 24\sin\varphi(\sin^2\varphi + 2\cos^2\varphi)$$

$$= 24\sin\varphi(1+\cos^2\varphi)$$

In the integral let $u=\cos\varphi$ and $du=-\sin\varphi\,d\varphi$

$$\iint \vec{F}\,d\vec{S} = \int_0^\pi \int_o^{2\pi} 24\sin\varphi(1+\cos^2\varphi)\,d\theta\,d\varphi = 48\pi\int_0^\pi \sin\varphi(1+\cos^2\varphi)\,d\varphi$$

$$= -48\pi\int_1^{-1}(1+u^2)\,du = -48\pi\left[u+\frac{u^3}{3}\right]_1^{-1} = -48\pi\left(-1+\frac{-1}{3}\right)+48\pi\left(1+\frac{1}{3}\right) = 128\pi$$

**5.** (8 points) Compute $\iint \vec{\nabla}\times\vec{F}\,d\vec{S}$ over the cone $z=\sqrt{x^2+y^2}$ for $z\le 3$ with normal pointing up and in, for the vector field $\vec{F}=(3y,-3x,6xy)$.

Note: The cone may be parametrized by $\vec{R}(r,\theta)=(r\cos\theta, r\sin\theta, r)$. Follow these steps:

$$\vec{e}_r = \big(\ \cos\theta,\ \ \sin\theta,\ 1\big)$$

$$\vec{e}_\theta = \big(-r\sin\theta,\ r\cos\theta,\ 0\big)$$

$$\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$

$$= (-r\cos\theta, -r\sin\theta, r)$$ which points up and in.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3y & -3x & 6xy \end{vmatrix}$$

$$= \hat{\imath}(6x) - \hat{\jmath}(6y) + \hat{k}(-3-3)$$

$$= (6x, -6y, -6)$$

$$\vec{\nabla}\times\vec{F}\big(\vec{R}(r,\theta)\big) = (6r\cos\theta, -6r\sin\theta, -6)$$

$$\vec{\nabla}\times\vec{F}\cdot\vec{N} = -6r^2\cos^2\theta + 6r^2\sin^2\theta - 6r$$

$$= -6r^2\cos 2\theta - 6r$$ since $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\iint \vec{\nabla}\times\vec{F}\,d\vec{S} = \int_0^{2\pi}\int_0^3 (-6r^2\cos 2\theta - 6r)\,dr\,d\theta = \int_0^{2\pi}\Big[-2r^3\cos 2\theta - 3r^2\Big]_{r=0}^3 d\theta$$

$$= \int_0^{2\pi}(-54\cos 2\theta - 27)\,d\theta = \Big[-27\sin 2\theta - 27\theta\Big]_{\theta=0}^{2\pi} = -54\pi$$