1-9	/45
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MATH 251 Exam 1 Spring 2006

Sections 506 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. A fly travels along the path $\vec{r}(t) = (3\sin t, -4\sin t, 5\cos t)$.
Find the arc length traveled by the fly between $(3,-4,0)$ and $(0,0,-5)$.

a. $\frac{\pi}{2}$

b. π

c. $\frac{3\pi}{2}$

d. 2π

e. $\frac{5\pi}{2}$ Correct Choice

The start is when $(3\sin t, -4\sin t, 5\cos t) = (3,-4,0)$ or $t = \frac{\pi}{2}$.

The finish is when $(3\sin t, -4\sin t, 5\cos t) = (0,0,-5)$ or $t = \pi$.

$\vec{v} = (3\cos t, -4\cos t, -5\sin t) \qquad |\vec{v}| = \sqrt{9\cos^2 t + 16\cos^2 t + 25\sin^2 t} = 5$

$L = \int_{\pi/2}^{\pi} |\vec{v}|\,dt = \int_{\pi/2}^{\pi} 5\,dt = \Big[5t\Big]_{\pi/2}^{\pi} = \frac{5\pi}{2}$

2. Find the unit binormal $\hat{B}$ to the curve $\vec{r}(t) = (3\sin t, -4\sin t, 5\cos t)$.

a. $\left(\frac{3}{5}, \frac{-4}{5}, 0\right)$

b. $\left(\frac{3}{5}, \frac{4}{5}, 0\right)$

c. $\left(\frac{4}{5}, \frac{-3}{5}, 0\right)$

d. $\left(\frac{4}{5}, \frac{3}{5}, 0\right)$ Correct Choice

e. $\left(\frac{-3}{5}, \frac{4}{5}, 0\right)$

$\vec{v} = (3\cos t, -4\cos t, -5\sin t) \qquad \vec{a} = (-3\sin t, 4\sin t, -5\cos t)$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3\cos t & -4\cos t & -5\sin t \\ -3\sin t & 4\sin t & -5\cos t \end{vmatrix}$$

$$= \hat{\imath}(20\cos^2 t + 20\sin^2 t) - \hat{\jmath}(-15\cos^2 t - 15\sin^2 t) + \hat{k}(12\cos t\sin t - 12\cos t\sin t) = (20, 15, 0)$$

$|\vec{v} \times \vec{a}| = \sqrt{400 + 225} = \sqrt{625} = 25 \qquad \hat{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$

3. Find the vector projection of the vector $\vec{u} = \langle 1, -2, 2 \rangle$ onto the vector $\vec{v} = \langle -1, 1, 1 \rangle$.

a. $\left\langle \frac{-1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$

b. $\left\langle \frac{1}{3}, \frac{-1}{3}, \frac{-1}{3} \right\rangle$ Correct Choice

c. $\left\langle \frac{1}{9}, \frac{-1}{9}, \frac{-1}{9} \right\rangle$

d. $\left\langle \frac{-1}{9}, \frac{2}{9}, \frac{-2}{9} \right\rangle$

e. $\left\langle \frac{1}{9}, \frac{-2}{9}, \frac{2}{9} \right\rangle$

$|\vec{v}|^2 = 1 + 1 + 1 = 3 \qquad \vec{u} \cdot \vec{v} = -1 - 2 + 2 = -1$

$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} = \frac{-1}{3}\langle -1, 1, 1 \rangle = \left\langle \frac{1}{3}, \frac{-1}{3}, \frac{-1}{3} \right\rangle$

4. Find the plane parallel to $4x - 3y + 2z = 5$ which passes through the point $P = (1, 2, 3)$. What is the z-intercept of this plane?

a. 2 Correct Choice

b. 4

c. 6

d. 8

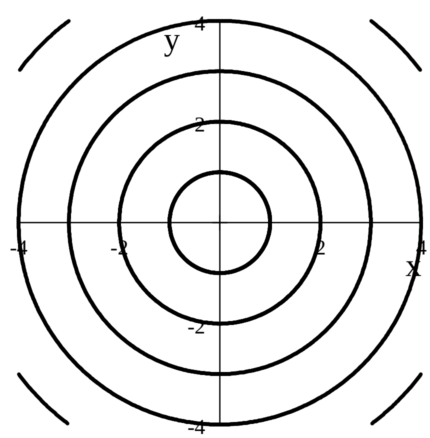
e. 10

$\vec{N} \cdot X = 4x - 3y + 2z$ So $\vec{N} = \langle 4, -3, 2 \rangle$, and $\vec{N} \cdot P = 4 \cdot 1 - 3 \cdot 2 + 2 \cdot 3 = 4$.

So the plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $4x - 3y + 2z = 4$ or $z = -2x + \frac{3}{2}y + 2$.

So the z-intercept is 2.

5. The plot at the right is the contour plot of which function?



a. The hyperbolic paraboloid $z = x^2 - y^2$

b. The hyperbolic paraboloid $z = xy$

c. The hyperboloid $z = \sqrt{1 + x^2 + y^2}$ Correct Choice

d. The cone $z = \sqrt{(x-1)^2 + y^2}$

e. The elliptic paraboloid $z = x^2 + (y-1)^2$

The level curves of $z = x^2 - y^2$ and $z = xy$ are hyperbolas.

The level curves of $z = \sqrt{(x-1)^2 + y^2}$ are circles centered at $(1, 0)$.

The level curves of $z = x^2 + (y-1)^2$ are circles centered at $(0, 1)$.

The level curves of $z = \sqrt{1 + x^2 + y^2}$ are circles centered at $(0, 0)$ as in the figure.

6. If $r = \sqrt{x^2+y^2}$, find $\frac{\partial^2 r}{\partial x \partial y}(3,4)$.

a. $\frac{12}{5}$

b. $\frac{12}{25}$

c. $\frac{12}{125}$

d. $\frac{-12}{125}$ Correct Choice

e. $\frac{-12}{5}$

$\frac{\partial r}{\partial y} = \frac{x}{\sqrt{x^2+y^2}} = x(x^2+y^2)^{-1/2}$ $\frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{2}x(x^2+y^2)^{-3/2}2y = \frac{-xy}{(x^2+y^2)^{3/2}}$ $\frac{\partial^2 r}{\partial x \partial y}(3,4) = \frac{-12}{125}$

7. The graph of a function $z = f(x,y)$ passes through the point $(2,4,8)$ and has $f_x(2,4) = -3$ and $f_y(2,4) = 5$. Use the linear approximation to approximate $f(2.2,3.9)$

a. -1.1

b. 1.1

c. 6.9 Correct Choice

d. 7.9

e. 9.1

$f_{\tan}(x,y) = f(2,4) + f_x(2,4)(x-2) + f_y(2,4)(y-4) = 8 - 3(x-2) + 5(y-4)$

$f(2.2,3.9) \approx f_{\tan}(2.2,3.9) = 8 - 3(2.2-2) + 5(3.9-4) = 8 - 3(.2) + 5(-.1) = 6.9$

8. Duke Skywater is travelling through the galaxy. At the present time he is at the point with galactic coordinates $P = (40,25,53)$ (in lightyears), and his velocity is $\vec{v} = (.1,-.2,.3)$ (in lightyears /year). He measures the polaron density to be $U = 4300$ polarons/cm^3 and its gradient to be $\vec{\nabla}U = (3,2,1)$ polarons/cm^3/lightyear. Find the rate $\frac{dU}{dt}$ at which Duke sees the polaron density changing (in polarons/cm^3/year).

a. 0.2 Correct Choice

b. 1.0

c. 223

d. 223.2

e. 224

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} = \vec{\nabla}U \cdot \vec{v} = (3,2,1)\cdot(.1,-.2,.3) = .3 - .4 + .3 = .2$$

9. The density of carbon monoxide in a room is given by $\delta = z\ln(9+x^2-y^2)$. If you start at the point $(3,4,2)$, in what direction should you move to ***decrease*** the carbon monoxide density as fast as possible?

a. $(-6,-8,-\ln 2)$

b. $(-6,8,-\ln 2)$ Correct Choice

c. $(6,-8,-\ln 2)$

d. $(-6,8,\ln 2)$

e. $(6,-8,\ln 2)$

$\vec{\nabla}\delta = \left(\frac{2xz}{9+x^2-y^2}, \frac{-2yz}{9+x^2-y^2}, \ln(9+x^2-y^2)\right)$ $\vec{\nabla}\delta(3,4,2) = \left(\frac{12}{2}, \frac{-16}{2}, \ln 2\right) = (6,-8,\ln 2)$

The density *decreases* fastest in the direction $-\vec{\nabla}\delta = (-6,8,-\ln 2)$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. Find the equation of the plane tangent to each of the following surfaces:

a. (10 points) $z = f(x,y) = xe^{xy}$ at $(x,y) = (2,0)$

$f = xe^{xy}$ $f_x = e^{xy} + xye^{xy}$ $fy = x^2e^{xy}$

$f(2,0) = 2$ $f_x(2,0) = 1$ $f_y(2,0) = 4$

$z = f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0) = 2 + 1(x-2) + 4(y) = x + 4y$

$\boxed{x + 4y - z = 0}$

b. (10 points) $xe^{yz} + ye^{xz} = 5$ at $(x,y,z) = (2,3,0)$

Let $f(x,y,z) = xe^{yz} + ye^{xz}$. $\vec{\nabla}f = (e^{yz} + yze^{xz}, xze^{yz} + e^{xz}, xye^{yz} + xye^{xz})$

$P = (2,3,0)$ $\vec{N} = \vec{\nabla}f(2,3,0) = (1,1,12)$

$\vec{N} \cdot X = \vec{N} \cdot P$ $1x + 1y + 12z = 1 \cdot 2 + 1 \cdot 3$

$\boxed{x + y + 12z = 5}$

11. $(15$ points$)$ The radius of a cylinder is currently $r = 5$ cm and its height is $h = 10$ cm.

Its radius is increasing at $\frac{dr}{dt} = 0.3\frac{\text{cm}}{\text{sec}}$ and its height is decreasing at $\frac{dh}{dt} = -0.2\frac{\text{cm}}{\text{sec}}$.

Is its volume increasing or decreasing and at what rate?

$V = \pi r^2 h$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt} = 2\pi(5)(10)(0.3) + \pi(5)^2(-0.2) = \pi(30-5) = 25\pi$$

V is increasing.

12. $(10$ points$)$ If $z = x^2y^3$ where $x = x(u,v)$ and $y = y(u,v)$ satisfy

$$x(3,4) = 2 \qquad \frac{\partial x}{\partial u}(3,4) = 5 \qquad \frac{\partial x}{\partial v}(3,4) = 6$$

$$y(3,4) = 1 \qquad \frac{\partial y}{\partial u}(3,4) = 7 \qquad \frac{\partial y}{\partial v}(3,4) = 8$$

Find $\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(3,4)}$.

$$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(3,4)} = \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(2,1)}\left.\frac{\partial x}{\partial v}\right|_{(u,v)=(3,4)} + \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(2,1)}\left.\frac{\partial y}{\partial v}\right|_{(u,v)=(3,4)}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(2,1)} = 2xy^3|_{(x,y)=(2,1)} = 4$$

$$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(2,1)} = 3x^2y^2|_{(x,y)=(2,1)} = 12$$

$$\left.\frac{\partial z}{\partial v}\right|_{(u,v)=(3,4)} = 4 \bullet 6 + 12 \bullet 8 = 24 + 96 = 120$$

13. $(15$ points$)$ Find the point on the ellipsoid $\frac{x^2}{16} + \frac{y^2}{4} + z^2 = 3$ at which the function $f = -x - 2y + 4z$ is a minimum.

You may use any method but Lagrange multipliers is easier than eliminating a variable.

METHOD 1: Lagrange Multipliers:

Let $g = \frac{x^2}{16} + \frac{y^2}{4} + z^2$. Then the constraint is $g = 3$.

$\vec{\nabla} f = (-1,-2,4) \qquad \vec{\nabla} g = \left(\frac{x}{8}, \frac{y}{2}, 2z\right)$

The Lagrange equations $\vec{\nabla} f = \lambda\vec{\nabla} g$ become:

$-1 = \lambda\frac{x}{8}, \quad -2 = \lambda\frac{y}{2}, \quad 4 = \lambda 2z$ or $x = \frac{-8}{\lambda}, \quad y = \frac{-4}{\lambda}, \quad z = \frac{2}{\lambda}$

From the constraint,

$3 = \frac{x^2}{16} + \frac{y^2}{4} + z^2 = \frac{64}{16\lambda^2} + \frac{16}{4\lambda^2} + \frac{4}{\lambda^2} = \frac{12}{\lambda^2}$ So $\lambda = \pm 2$

If $\lambda = 2$, then $x = -4$, $y = -2$, $z = 1$ and $f = -x - 2y + 4z = 4 + 4 + 4 = 12$

If $\lambda = -2$, then $x = 4$, $y = 2$, $z = -1$ and $f = -x - 2y + 4z = -4 - 4 - 4 = -12$

So the minimum occurs at $(x,y,z) = (4,2,-1)$

13. $\left(15\text{ points}\right)$ Find the point on the ellipsoid $\frac{x^2}{16}+\frac{y^2}{4}+z^2=3$ at which the function $f=-x-2y+4z$ is a minimum.

You may use any method but Lagrange multipliers is easier than eliminating a variable.

METHOD 2: Eliminate a Variable:

$z=\pm\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$

Case 1: If $z=\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$ then $f=-x-2y+4\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$

$f_x=-1+\frac{4}{2\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}}\left(\frac{-2x}{16}\right)=0 \qquad \frac{-x}{4\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}}=1 \qquad -x=4\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$

$x^2=48-x^2-4y^2 \qquad 2x^2+4y^2=48 \qquad x^2+2y^2=24$ (1)

$f_y=-2+\frac{4}{2\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}}\left(\frac{-2y}{4}\right)=0 \qquad \frac{-y}{\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}}=2 \qquad -y=2\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$

$y^2=12-\frac{x^2}{4}-y^2 \qquad \frac{x^2}{4}+2y^2=12 \qquad x^2+8y^2=48$ (2)

Subtract (2)-(1): $6y^2=24 \qquad y=\pm2 \qquad x^2=24-2y^2=24-8=16 \qquad x=\pm4$

$z=\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}=\sqrt{3-\frac{16}{16}-\frac{4}{4}}=1$

Critical points: $(4,2,1),(4,-2,1),(-4,2,1),(-4,-2,1)$

$f=-x-2y+4z \qquad f(4,2,1)=-4, \qquad f(4,-2,1)=4 \qquad ,f(-4,2,1)=4, \qquad f(-4,-2,1)=12$

Case 2: If $z=-\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$ then $f=-x-2y-4\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$

$f_x=-1-\frac{4}{2\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}}\left(\frac{-2x}{16}\right)=0 \qquad \frac{x}{4\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}}=1 \qquad x=4\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$

$x^2=48-x^2-4y^2 \qquad 2x^2+4y^2=48 \qquad x^2+2y^2=24$ (1)

$f_y=-2-\frac{4}{2\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}}\left(\frac{-2y}{4}\right)=0 \qquad \frac{y}{\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}}=2 \qquad y=2\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}$

$y^2=12-\frac{x^2}{4}-y^2 \qquad \frac{x^2}{4}+2y^2=12 \qquad x^2+8y^2=48$ (2)

Subtract (2)-(1): $6y^2=24 \qquad y=\pm2 \qquad x^2=24-2y^2=24-8=16 \qquad x=\pm4$

$z=-\sqrt{3-\frac{x^2}{16}-\frac{y^2}{4}}=-\sqrt{3-\frac{16}{16}-\frac{4}{4}}=-1$

Critical points: $(4,2,-1),(4,-2,-1),(-4,2,-1),(-4,-2,-1)$

$f=-x-2y+4z \qquad f(4,2,-1)=-12, \qquad f(4,-2,-1)=-4, \qquad f(-4,2,-1)=-4, \qquad f(-4,-2,-1)=4$

So the minimum occurs at $(x,y,z)=(4,2,-1)$