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1-12	/60
13	/20
14	/10
15	/15
Total	/105

MATH 251 Final Exam Spring 2006

Sections 506 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Let L be the line $\vec{r}(t) = (6+7t, -3-4t, 5-2t)$. Find the equation of the plane perpendicular to L that contains the point $(3,-5,2)$.

a. $3(x-7) - 5(y+4) + 2(z+2) = 0$

b. $3(x-7) + 5(y+4) + 2(z+2) = 0$

c. $7(x-3) - 4(y+5) - 2(z-2) = 0$ Correct Choice

d. $7(x-3) + 4(y+5) - 2(z-2) = 0$

e. $6(x-3) - 3(y+5) + 5(z-2) = 0$

The normal vector to the plane is the tangent vector to the line: $\vec{N} = \vec{v} = (7,-4,-2)$

The plane passes thru $P = (3,-5,2)$. So its equation is $\vec{N} \cdot (X-P) = 0$ or

$7(x-3) - 4(y+5) - 2(z-2) = 0$

2. Find the equation of the plane tangent to the graph of $z = x\sin y$ at $(x,y) = \left(2, \frac{\pi}{3}\right)$.

a. $z = \frac{1}{2}x + \sqrt{3}y - \frac{\pi}{\sqrt{3}} + \sqrt{3} - 1$

b. $z = \frac{1}{2}x + \sqrt{3}y - \frac{\pi}{\sqrt{3}} + \sqrt{3}$

c. $z = \frac{1}{2}x + \sqrt{3}y + \sqrt{3} - 1$

d. $z = \frac{\sqrt{3}}{2}x + y - \frac{\pi}{3}$ Correct Choice

e. $z = \frac{\sqrt{3}}{2}x + y + \sqrt{3}$

Let $f = x\sin y$. Then $f\left(2, \frac{\pi}{3}\right) = 2\sin\frac{\pi}{3} = \sqrt{3}$.

$f_x = \sin y$, $f_y = x\cos y$, $f_x\left(2, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $f_y\left(2, \frac{\pi}{3}\right) = 2\cos\frac{\pi}{3} = 1$

The tangent plane is

$$z = f_{\tan}(x,y) = f\left(2, \frac{\pi}{3}\right) + f_x\left(2, \frac{\pi}{3}\right)(x-2) + f_y\left(2, \frac{\pi}{3}\right)\left(y - \frac{\pi}{3}\right)$$

$$= \sqrt{3} + \frac{\sqrt{3}}{2}(x-2) + 1\left(y - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}x + y - \frac{\pi}{3}$$

3. Find the equation of the line perpendicular to the surface $xy + z^2 = 6$ at the point $(1,2,2)$.

a. $x = 2 + t, y = -1 + 2t, z = 4 + 2t$

b. $x = 2 + t, y = -1 - 2t, z = 4 + 2t$

c. $x = 2 + t, y = 1 + 2t, z = 4 + 2t$

d. $x = 1 + 2t, y = 2 - t, z = 2 + 4t$

e. $x = 1 + 2t, y = 2 + t, z = 2 + 4t$ Correct Choice

Let $f = xy + z^2$. The gradient of f is perpenducular to its level sets: $\vec{\nabla} f = (y, x, 2z)$

The tangent to the line is the normal at $P = (1,2,2)$: $\vec{v} = \vec{N} = \vec{\nabla} f\Big|_{(1,2,2)} = (2,1,4)$

So the line is $X = P + t\vec{v} = (1,2,2) + t(2,1,4) = (1 + 2t, 2 + t, 2 + 4t)$

4. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2 - 2y}{y - xy} =$

a. -2 Correct Choice

b. 0

c. 1

d. 2

e. Does Not Exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2 - 2y}{y - xy} = \lim_{(x,y)\to(0,0)} \frac{x^2y - 2}{1 - x} = -2$$

5. The radius and height of a cylinder are currently $r = 10$ cm and $h = 6$ cm. If the radius is increasing at $\dfrac{dr}{dt} = 2\,\dfrac{\text{cm}}{\text{min}}$ and the volume is increasing at $\dfrac{dV}{dt} = 40\pi\,\dfrac{\text{cm}^3}{\text{min}}$, is the height increasing or decreasing and at what rate?

a. decreasing at $2\,\dfrac{\text{cm}}{\text{min}}$ Correct Choice

b. decreasing at $\dfrac{4}{5}\,\dfrac{\text{cm}}{\text{min}}$

c. increasing at $2\,\dfrac{\text{cm}}{\text{min}}$

d. increasing at $\dfrac{4}{5}\,\dfrac{\text{cm}}{\text{min}}$

e. The height is constant.

$$V = \pi r^2 h \qquad \frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$$

At present, $40\pi = 2\pi 60 \cdot 2 + \pi 100\dfrac{dh}{dt}$ So $100\dfrac{dh}{dt} = 40 - 240 = -200$ or $\dfrac{dh}{dt} = -2$

6. Han Duet is flying the Millenium Eagle through a radion field with density $\rho = z(x+y)$. He is currently located at $(-4,3,5)$ in galactic coordinates. In what direction should he fly to decrease the radion density as fast as possible?

a. $(-5,5,1)$

b. $(-5,-5,1)$ Correct Choice

c. $(5,5,-1)$

d. $(28,-21,35)$

e. $(-28,21,-35)$

$\vec{\nabla}\rho = (z,z,x+y) \qquad \vec{\nabla}\rho\big|_{(-4,3,5)} = (5,5,-1)$

The gradient points in the direction of maximum increase of the density.

So the direction of maximum decrease is $-\vec{\nabla}\rho\big|_{(-4,3,5)} = (-5,-5,1)$.

7. Han Duet is flying the Millenium Eagle through a radion field with density $\rho = z(x+y)$. He is currently located at $(-4,3,5)$ in galactic coordinates and has velocity $\vec{v} = (0.2,-0.1,0.3)$. What does he see as the time rate of change of the radion density?

a. 0.2 Correct Choice

b. -0.2

c. 1.2

d. -1.2

e. 0.4

$$\frac{d\rho}{dt} = \vec{v}\cdot\vec{\nabla}\rho = (0.2,-0.1,0.3)\cdot(5,5,-1) = 1-.5-.3 = .2$$

8. Find the volume below $z = 2x^2y$ above the region in the xy-plane bounded by $y = 0$, $y = x^2$ and $x = 2$.

a. $\dfrac{32}{5}$

b. $\dfrac{32}{3}$

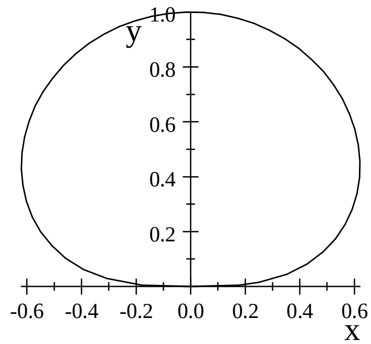
c. $\dfrac{128}{7}$ Correct Choice

d. 32

e. $\dfrac{512}{9}$

$$V = \int_0^2\int_0^{x^2} 2x^2y\,dy\,dx = \int_0^2\Big[x^2y^2\Big]_{y=0}^{x^2}dx = \int_0^2 x^6\,dx = \frac{x^7}{7}\Big|_{x=0}^{2} = \frac{128}{7}$$

9. The graph of the polar curve $r = \sqrt{\sin(\theta)}$ is shown at the right. Find the area enclosed.



a. 1.2

b. 1.0 Correct Choice

c. 0.8

d. $\frac{\pi}{3}$

e. $\frac{\pi}{4}$

$$A = \iint 1\,dA = \int_0^\pi \int_0^{\sqrt{\sin\theta}} r\,dr\,d\theta = \int_0^\pi \left[\frac{r^2}{2} \right]_{r=0}^{\sqrt{\sin\theta}} d\theta = \frac{1}{2}\int_0^\pi \sin\theta\,d\theta = -\frac{1}{2}\cos\theta \Big|_{\theta=0}^{\pi} = \frac{1}{2} - -\frac{1}{2} = 1$$

10. Compute $\int_{(-1,0,-1)}^{(2,0,8)} \vec{F} \cdot d\vec{s}$ where $\vec{F} = (3x^2, 2y, 1)$ along the curve

$\vec{r}(t) = (t\cos(\pi t), t^2\sin(\pi t), t^3\cos(\pi t))$.

HINT: Note $\vec{F} = \vec{\nabla} f$ where $f = x^3 + y^2 + z$.

a. 8

b. 9

c. 12

d. 15

e. 18 Correct Choice

By the Fundamental Theorem of Calculus for Curves,

$$\int_{(-1,0,-1)}^{(2,0,8)} \vec{F} \cdot d\vec{s} = \int_{(-1,0,-1)}^{(2,0,8)} \vec{\nabla} f \cdot d\vec{s} = f(2,0,8) - f(-1,0,-1) = (8+8) - (-1-1) = 18$$

11. Compute $\oint (\ln x - 3xe^y)\,dx + (x^2 e^y)\,dy$ along the closed curve which travels along the straight line from $(0,0)$ to $(1,0)$, along the straight line from $(1,0)$ to $(1,1)$ and along $y = x^2$ from $(1,1)$ to $(0,0)$.

a. $5 - 5e$

b. $5e - 5$

c. $5 - \frac{5}{2}e$

d. $\frac{5}{2}e - 5$ Correct Choice

e. Diverges

By Green's Theorem,

$$\oint (\ln x - 3xe^y)\,dx + (x^2 e^y)\,dy = \iint \left[\frac{\partial}{\partial x}(x^2 e^y) - \frac{\partial}{\partial y}(\ln x - 3xe^y) \right] dx\,dy$$

$$= \iint [(2xe^y) - (-3xe^y)]\,dx\,dy = \int_0^1 \int_0^{x^2} 5xe^y\,dy\,dx = \int_0^1 \left[5xe^y \right]_{y=0}^{x^2} dx = \int_0^1 \left(5xe^{x^2} - 5x \right) dx$$

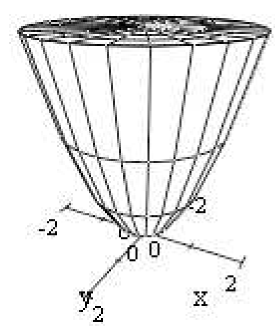
$$= \left[\frac{5}{2}e^{x^2} - \frac{5}{2}x^2 \right]_0^1 = \left(\frac{5}{2}e - \frac{5}{2} \right) - \left(\frac{5}{2} \right) = \frac{5}{2}e - 5$$

12. Compute $\iint_{\partial P} \vec{F} \cdot d\vec{S}$ over the complete surface of the solid paraboloid

$$x^2 + y^2 \le z \le 4$$

with outward normal, for the vector field

$$\vec{F} = (x^3, y^3, z)$$



a. $\frac{16\pi}{3}$

b. $\frac{32\pi}{3}$

c. 16π

d. 32π Correct Choice

e. 48π

Apply Gauss' Theorem: $\vec{\nabla} \cdot F = 3x^2 + 3y^2 + 1 = 3r^2 + 1$ $\quad dV = r\,dr\,d\theta\,dz$

$$\iint_{\partial P} \vec{F} \cdot d\vec{S} = \iiint_P \vec{\nabla} \cdot F\,dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (3r^2 + 1) r\,dz\,dr\,d\theta = 2\pi \int_0^2 [(3r^3 + r)z]_{z=r^2}^4\,dr = 2\pi \int_0^2 (3r^3 + r)(4 - r^2)$$

$$= 2\pi \int_0^2 (12r^3 - 3r^5 + 4r - r^3)\,dr = 2\pi \left[3r^4 - \frac{r^6}{2} + 2r^2 - \frac{r^4}{4} \right]_0^2 = 2\pi(48 - 32 + 8 - 4) = 40\pi$$

NO CORRECT ANSWER! Problem will be thrown out. Everyone gets the 5 points.

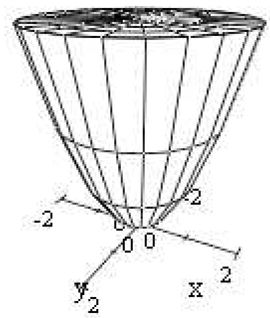
Intended problem 12:

Compute $\iint_{\partial P} \vec{F} \cdot d\vec{S}$ over the complete surface of the solid paraboloid

$$x^2 + y^2 \le z \le 4$$

with outward normal, for the vector field

$$\vec{F} = (x^3, y^3, x + y)$$



a. $\frac{16\pi}{3}$

b. $\frac{32\pi}{3}$

c. 16π

d. 32π Correct Choice

e. 48π

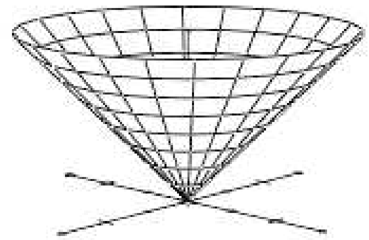
Apply Gauss' Theorem: $\vec{\nabla} \cdot F = 3x^2 + 3y^2 = 3r^2$ $\quad dV = r\,dr\,d\theta\,dz$

$$\iint_{\partial P} \vec{F} \cdot d\vec{S} = \iiint_P \vec{\nabla} \cdot F\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 3r^2 r\,dz\,dr\,d\theta = 2\pi\int_0^2 [3r^3 z]_{z=r^2}^4\,dr = 2\pi\int_0^2 3r^3(4 - r^2)\,dr$$

$$= 2\pi\int_0^2 (12r^3 - 3r^5)\,dr = 2\pi\left[3r^4 - \frac{r^6}{2}\right]_0^2 = 2\pi(48 - 32) = 32\pi$$

Work Out: (Points indicated. Part credit possible.)

13. $\left(20\text{ points}\right)$ Verify Stokes' Theorem



$$\iint_C \vec{\nabla}\times\vec{F}\bullet d\vec{S} = \oint_{\partial C} \vec{F}\bullet d\vec{s}$$

for the vector field $\vec{F} = (-yz^2, xz^2, z^3)$ and the cone $z = \sqrt{x^2+y^2}$ for $z \le 3$ oriented down and out.

Be sure to check and explain the orientations.

Use the following steps:

a. The conical surface may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.
Compute the surface integral:
Successively find: $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$, $\vec{\nabla}\times\vec{F}$, $\vec{\nabla}\times\vec{F}\left(\vec{R}(r,\theta)\right)$, $\iint_C \vec{\nabla}\times\vec{F}\bullet d\vec{S}$

$$\begin{array}{rccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 1) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$$

$\vec{N}$ points up and in. Reverse it: $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz^2, & xz^2, & z^3 \end{vmatrix} = \hat{\imath}(0-2xz) - \hat{\jmath}(0--2yz) + \hat{k}(z^2--z^2) = (-2xz, -2yz, 2z^2)$$

$$\vec{\nabla}\times\vec{F}\left(\vec{R}(r,\theta)\right) = (-2r^2\cos\theta, -2r^2\sin\theta, 2r^2)$$

$$\vec{\nabla}\times\vec{F}\bullet\vec{N} = -2r^3\cos^2\theta - 2r^3\sin^2\theta - 2r^3 = -4r^3$$

$$\iint_C \vec{\nabla}\times\vec{F}\bullet d\vec{S} = \iint_C \vec{\nabla}\times\vec{F}\bullet\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 -4r^3\,dr\,d\theta = 2\pi[-r^4]_{r=0}^3 = -162\pi$$

Problem Continued

b. Parametrize the boundary circle ∂C and compute the line integral.
Successively find: $\vec{r}(\theta)$, $\vec{v}(\theta)$, $\vec{F}(\vec{r}(\theta))$, $\oint_{\partial C} \vec{F} \cdot d\vec{s}$.

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 3)$

$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$

By the right hand rule the upper curve must be traversed clockwise but $\vec{v}$ points counterclockwise. So reverse $\vec{v}$: $\vec{v}(\theta) = (3\sin\theta, -3\cos\theta, 0)$

$\vec{F}(\vec{r}(\theta)) = (-yz^2, xz^2, z^3) = (-27\sin\theta, 27\cos\theta, 27)$

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -81\sin^2\theta - 81\cos^2\theta\, d\theta = \int_0^{2\pi} -81\, d\theta = -162\pi$$

They agree!

14. (10 points) Find all critical points of $f(x,y) = xy - \frac{1}{3}x^3 - y^2$ and classify each of them as either a local minimum, a local maximum or a a saddle point. Justify your answers.

$$\left.\begin{aligned} f_x &= y - x^2 = 0 \\ f_y &= x - 2y = 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} y &= x^2 \\ x &= 2y = 2x^2 \end{aligned} \quad \Rightarrow \quad x(1-2x) = 0 \quad \Rightarrow \quad x = 0, \frac{1}{2}$$

If $x = 0$ then $y = 0$. If $x = \frac{1}{2}$ then $y = \frac{1}{4}$.

So the critical points are $(0,0)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$.

Apply the Second Derivative Test:

$f_{xx} = -2x \qquad f_{yy} = -2 \qquad f_{xy} = 1 \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 4x - 1$

$D(0,0) = -1 < 0 \quad \Rightarrow \quad (0,0)$ is a saddle

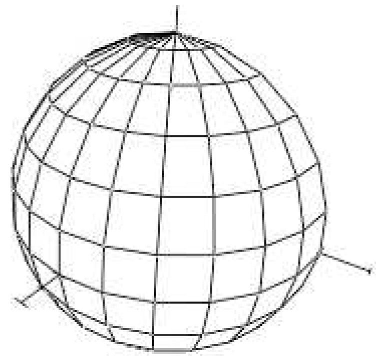
$D\left(\frac{1}{2}, \frac{1}{4}\right) = 4\left(\frac{1}{2}\right) - 1 = 1 > 0 \quad \& \quad f_{xx}\left(\frac{1}{2}, \frac{1}{4}\right) = -2\left(\frac{1}{2}\right) = -1 < 0$

$\Rightarrow \quad \left(\frac{1}{2}, \frac{1}{4}\right)$ is a local maximum

15. (15 points) Find the mass and z-component of the center of mass of the solid hemisphere

$$0 \le x \le \sqrt{4 - y^2 - z^2}$$

if the density is given by $\delta = 3 + z$.



In spherical coordinates, $z = \rho\cos\varphi$, $\delta = 3 + z = 3 + \rho\cos\varphi$ and $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

$$M = \iiint \delta \, dV = \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^2 (3 + \rho\cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \pi \int_0^{\pi} \int_0^2 (3\rho^2 + \rho^3 \cos\varphi) \sin\varphi \, d\rho \, d\varphi$$

$$= \pi \int_0^{\pi} \left[\rho^3 + \frac{\rho^4}{4} \cos\varphi \right]_{\rho=0}^{2} \sin\varphi \, d\varphi = \pi \int_0^{\pi} (8 + 4\cos\varphi) \sin\varphi \, d\varphi \qquad u = \cos\varphi \qquad du = -\sin\varphi \, d\varphi$$

$$= -\pi \int_1^{-1} (8 + 4u) \, du = -\pi [8u + 2u^2]_1^{-1} = -\pi(-6 - 10) = 16\pi$$

$$M_{xy} = \iiint z\delta \, dV = \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^2 \rho\cos\varphi(3 + \rho\cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \pi \int_0^{\pi} \int_0^2 (3\rho^3 + \rho^4 \cos\varphi) \cos\varphi \sin\varphi \, d\rho$$

$$= \pi \int_0^{\pi} \left[\frac{3\rho^4}{4} + \frac{\rho^5}{5} \cos\varphi \right]_{\rho=0}^{2} \cos\varphi \sin\varphi \, d\varphi = \pi \int_0^{\pi} \left(12 + \frac{32}{5} \cos\varphi \right) \cos\varphi \sin\varphi \, d\varphi$$

$$= -\pi \int_1^{-1} \left(12 + \frac{32}{5} u \right) u \, du = -\pi \left[6u^2 + \frac{32}{15} u^3 \right]_1^{-1} = -\pi \left[\left(6 - \frac{32}{15} \right) - \left(6 + \frac{32}{15} \right) \right] = \frac{64\pi}{15}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{64\pi}{15 \cdot 16\pi} = \frac{4}{15}$$