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MATH 251 Exam 1 Fall 2006

Sections 507 Solutions P. Yasskin

1-11	/55	14	/12
12	/12	15	/12
13	/12	16	/12
Total	/103		

Multiple Choice: (5 points each. No part credit.)

1. The vertices of a triangle are $P = (3,4,-5)$, $Q = (3,5,-4)$ and $R = (5,2,-5)$.
Find the angle at P.

a. $90°$

b. $120°$ Correct Choice

c. $135°$

d. $150°$

e. $180°$

$$\overrightarrow{PQ} = Q - P = \langle 0,1,1\rangle \qquad \overrightarrow{PR} = R - P = \langle 2,-2,0\rangle \qquad \left|\overrightarrow{PQ}\right| = \sqrt{2} \qquad \left|\overrightarrow{PR}\right| = \sqrt{8} \qquad \overrightarrow{PQ}\cdot\overrightarrow{PR} = -2$$

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|} = \frac{-2}{\sqrt{2}\sqrt{8}} = \frac{-1}{2} \qquad \theta = 120°$$

2. Find the volume of the parallelepiped with edge vectors:
$\vec{a} = \langle 4,1,2\rangle \quad \vec{b} = \langle 2,2,1\rangle \quad \vec{c} = \langle 1,3,0\rangle$

a. -3

b. 0

c. $\sqrt{3}$

d. 3 Correct Choice

e. 9

$$V = \left|\vec{a}\cdot\vec{b}\times\vec{c}\right| = \left\| \begin{matrix} 4 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 3 & 0 \end{matrix} \right\| = |0 + 1 + 12 - 4 - 12 - 0| = |-3| = 3$$

3. Consider the set of all points P whose distance from $(1,0,0)$ is 3 times its distance from $(-1,0,0)$. This set is a

a. sphere. Correct Choice

b. ellipsoid.

c. hyperboloid.

d. elliptic paraboloid.

e. hyperbolic paraboloid.

$\sqrt{(x-1)^2+y^2+z^2} = 3\sqrt{(x+1)^2+y^2+z^2}$ $\qquad (x-1)^2+y^2+z^2 = 9(x+1)^2+9y^2+9z^2$

$0 = 8x^2+20x+8y^2+8z^2+8 \qquad 0 = x^2+\frac{5}{2}x+y^2+z^2+1 = \left(x+\frac{5}{4}\right)^2+y^2+z^2-\frac{9}{16}$ sphere

4. For the curve $\vec{r}(t) = (\sin^2 t, \quad \cos^2 t, \quad \sin^2 t - \cos^2 t)$ which of the following is FALSE?

a. $\vec{v} = \langle 2\sin t\cos t, \quad -2\sin t\cos t, \quad 4\sin t\cos t\rangle$

b. $|\vec{v}| = \sqrt{24}\,\sin t\cos t$

c. $\hat{T} = \left\langle \frac{2}{\sqrt{24}}, \quad \frac{-2}{\sqrt{24}}, \quad \frac{4}{\sqrt{24}} \right\rangle$

d. $a_T = 0$ Correct Choice

e. $a_N = 0$

$\vec{v}$, $|\vec{v}|$, and $\hat{T}$ are correct by computation.

Since $\hat{T}$ is constant, its direction does not change and $a_N = 0$.

Since $|\vec{v}|$ is not constant, $a_T = \dfrac{d|\vec{v}|}{dt} \neq 0$.

5. For the curve $\vec{r}(t) = (\sin^2 t, \quad \cos^2 t, \quad \sin^2 t - \cos^2 t)$ compute the arc length between $\vec{r}(0) = (0, \quad 1, \quad -1)$ and $\vec{r}\left(\frac{\pi}{2}\right) = (1, \quad 0, \quad 1)$.

a. $\frac{1}{4}\sqrt{6}$

b. $\frac{1}{2}\sqrt{6}$

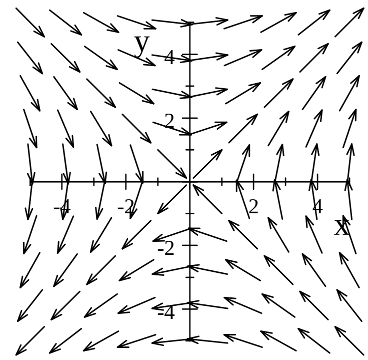
c. $\sqrt{6}$ Correct Choice

d. $2\sqrt{6}$

e. $4\sqrt{6}$

$$L = \int_0^{\pi/2} \sqrt{24}\,\sin t\cos t\,dt = \sqrt{24}\,\frac{\sin^2 t}{2}\Big|_0^{\pi/2} = \frac{1}{2}\sqrt{24} = \sqrt{6}$$

6. The plot at the right represents which vector field?



a. $\vec{A} = \langle x, y \rangle$

b. $\vec{B} = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$

c. $\vec{C} = \langle y, x \rangle$

d. $\vec{D} = \left\langle \dfrac{y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}} \right\rangle$ Correct Choice

e. $\vec{E} = \langle x+y, x-y \rangle$

The vectors all have the same length. So it must be one of the unit vector fields: $\vec{B}$ or $\vec{D}$.
$\vec{B}$ points radial. $\vec{D}$ is vertical on the x-axis and horizontal on the y-axis.

7. Describe the level surfaces of $f(x,y,z) = x^2 - y^2 - z^2$.

a. Elliptic Paraboloids

b. Elliptic and Hyperbolic Paraboloids

c. Hyperboloids of 1-sheet only

d. Hyperboloids of 2-sheets only

e. Hyperboloids of 1-sheet or 2-sheets Correct Choice

$x^2 - y^2 - z^2 = C$ is a hyperboloid with 2-sheets if $C > 0$, and 1-sheet if $C < 0$, and a cone if $C = 0$.

8. Find the plane tangent to the graph of $z = x e^{xy}$ at the point $(2,0)$. Its z-intercept is

a. 0 Correct Choice

b. 2

c. -2

d. 4

e. -4

$f = x e^{xy}$ $\qquad f(2,0) = 2$ $\qquad z = f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0)$

$f_x = e^{xy} + xy e^{xy}$ $\qquad f_x(2,0) = 1$ $\qquad = 2 + 1(x-2) + 4(y) = x + 4y$

$f_y = x^2 e^{xy}$ $\qquad f_y(2,0) = 4$ $\qquad$ The z-intercept is 0.

9. Find the plane tangent to the surface $xyz + z^2 = 28$ at the point $(4,3,2)$.

Its z-intercept is

a. 0

b. 5 Correct Choice

c. -5

d. 80

e. -80

$\vec{\nabla}F = \langle yz, xz, xy + 2z\rangle \qquad \vec{N} = \vec{\nabla}F(4,3,2) = \langle 6,8,16\rangle \qquad \vec{N}\cdot X = \vec{N}\cdot P$

$6x + 8y + 16z = 6\cdot 4 + 8\cdot 3 + 16\cdot 2 = 80 \qquad z = \frac{80}{16} - \frac{6}{16}x - \frac{8}{16}y = 5 - \frac{3}{8}x - \frac{1}{2}y$

The z-intercept is 5.

10. Find the line normal to the surface $xyz + z^2 = 28$ at the point $(4,3,2)$.

It intersects the xy-plane at

a. $(4,3,2)$

b. $(4,3,0)$

c. $\left(\frac{13}{4},2,0\right)$ Correct Choice

d. $\left(\frac{19}{4},4,4\right)$

e. $\left(\frac{19}{4},4,0\right)$

$\vec{\nabla}F = \langle yz, xz, xy + 2z\rangle \qquad \vec{N} = \vec{\nabla}F(4,3,2) = \langle 6,8,16\rangle \qquad X = P + t\vec{N}$

$(x,y,z) = (4,3,2) + t(6,8,16) = (4+6t, 3+8t, 2+16t)$

xy-plane is $z = 0$ or $2 + 16t = 0$ or $t = -\frac{1}{8}$

$(x,y,z) = \left(4 - \frac{3}{4}, 3 - 1, 2 - 2\right) = \left(\frac{13}{4},2,0\right)$

11. The salt concentration in a region of sea water is $\rho = xy^2z^3$. A swimmer is located at $(3,2,1)$.

In what direction should the swimmer swim to increase the salt concentration as fast as possible?

a. $\langle 4,-12,36\rangle$

b. $\langle -4,12,-36\rangle$

c. $\langle 4,12,36\rangle$ Correct Choice

d. $\langle -4,-12,-36\rangle$

e. $\langle 4,-12,-36\rangle$

$\vec{\nabla}\rho = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle = \langle 4,12,36\rangle$

Work Out: (12 points each. Part credit possible. Show all work.)

Do 4 of the following 5 problems. Cross out the one you do not want graded, here and on page 1. If you do not specify, #12 will be dropped.

12. Which of the following functions satisfy the Laplace equation $f_{xx} + f_{yy} = 0$? Show your work!

a. $f = x^2 + y^2$ NO

$f_{xx} + f_{yy} = 2 + 2 \neq 0$

b. $f = x^2 - y^2$ YES

$f_{xx} + f_{yy} = 2 - 2 = 0$

c. $f = x^3 + 3xy^2$ NO

$f_{xx} + f_{yy} = 6x + 6x \neq 0$

d. $f = x^3 - 3xy^2$ YES

$f_{xx} + f_{yy} = 6x - 6x = 0$

e. $f = e^{-x}\cos y + e^{-y}\cos x$ YES

$$f_{xx} + f_{yy} = (e^{-x}\cos y - e^{-y}\cos x) + (-e^{-x}\cos y + e^{-y}\cos x) = 0$$

f. $f = e^{-x}\cos y - e^{-y}\cos x$ YES

$$f_{xx} + f_{yy} = (e^{-x}\cos y + e^{-y}\cos x) + (-e^{-x}\cos y - e^{-y}\cos x) = 0$$

13. When two resistors with resistances R_1 and R_2 are connected in parallel, the net resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \text{or} \qquad R = \frac{R_1 R_2}{R_1 + R_2}.$$

If R_1 and R_2 are measured as $R_1 = 2 \pm 0.01$ ohms and $R_2 = 3 \pm 0.04$ ohms, then R can be calculated as $R = \frac{6}{5} \pm \Delta R$ ohms.

Use differentials to estimate the uncertainty ΔR in the computed value of R.

$$\Delta R = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2 = \frac{(R_1 + R_2)R_2 - R_1 R_2}{(R_1 + R_2)^2} dR_1 + \frac{(R_1 + R_2)R_1 - R_1 R_2}{(R_1 + R_2)^2} dR_2$$

$$= \frac{(R_2)^2}{(R_1 + R_2)^2} dR_1 + \frac{(R_1)^2}{(R_1 + R_2)^2} dR_2 = \frac{9}{25}(0.01) + \frac{4}{25}(0.04) = \frac{0.09 + 0.16}{25} = 0.01$$

14. The average of a function f on a curve $\vec{r}(t)$ is $f_{\text{ave}} = \dfrac{\int f\,ds}{\int ds}$.

Find the average of $f(x,y) = x^2$ on the circle $x^2 + y^2 = 9$.

HINTS: Parametrize the circle. $\sin^2 A = \dfrac{1-\cos(2A)}{2}$ $\cos^2 A = \dfrac{1+\cos(2A)}{2}$

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta)$ $\vec{v} = (-3\sin\theta, 3\cos\theta)$ $|\vec{v}| = \sqrt{9\sin^2\theta + 9\cos^2\theta} = 3$

$\int ds = \int_0^{2\pi} 3\,d\theta = 6\pi$ $f(r(t)) = (3\cos\theta)^2$

$$\int f\,ds = \int_0^{2\pi} 9\cos^2\theta\, 3\,d\theta = 27\int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\,d\theta = \frac{27}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{2\pi} = 27\pi$$

$f_{\text{ave}} = \dfrac{27\pi}{6\pi} = \dfrac{9}{2}$

15. A particle moves along the curve $\vec{r}(t) = (t^3, t^2, t)$ from $(1,1,1)$ to $(8,4,2)$ under the action of the force $\vec{F} = \langle z,y,x\rangle$. Find the work done.

$\vec{v} = \langle 3t^2, 2t, 1\rangle$ $\vec{F}(\vec{r}(t)) = \langle t, t^2, t^3\rangle$

$$W = \int_{(1,1,1)}^{(8,4,2)} \vec{F}\cdot d\vec{s} = \int_1^2 \vec{F}(\vec{r}(t))\cdot\vec{v}\,dt = \int_1^2 (3t^3 + 2t^3 + t^3)\,dt$$

$$= \int_1^2 6t^3\,dt = 6\frac{t^4}{4}\Big|_1^2 = \frac{3}{2}(16-1) = \frac{45}{2}$$

16. The pressure in an ideal gas is given by $P = k\rho T$ where k is a constant, ρ is the density and T is the temperature. At a certain instant, the measuring instruments are located at $r_o = (1,2,3)$ and moving with velocity $\vec{v} = \langle 4,5,6\rangle$ and acceleration $\vec{a} = \langle 7,8,9\rangle$. At that instant, the density and temperature are measured to be $\rho = 12$ and $T = 300$ and their gradients are $\vec{\nabla}\rho = \langle 0.6, 0.4, 0.2\rangle$ and $\vec{\nabla}T = \langle 2,1,4\rangle$.

Find $\dfrac{dP}{dt}$, the time rate of change of the pressure as seen by the instruments.

Your answer may depend on k.

HINTS: The pressure, P is a function of density, ρ, and temperature, T, which are functions of the position coordinates, (x,y,z), which are functions of time, t. Use the chain rule.

$\dfrac{\partial P}{\partial \rho} = kT = k300$ $\dfrac{\partial P}{\partial T} = k\rho = k12$

$$\frac{dP}{dt} = \frac{\partial P}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt} = \frac{\partial P}{\partial\rho}\left(\frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt}\right) + \frac{\partial P}{\partial T}\left(\frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}\right)$$

$$= \frac{\partial P}{\partial\rho}\left(\vec{v}\cdot\vec{\nabla}\rho\right) + \frac{\partial P}{\partial T}\left(\vec{v}\cdot\vec{\nabla}T\right) = k300\left(\langle 4,5,6\rangle\cdot\langle 0.6,0.4,0.2\rangle\right) + k12\left(\langle 4,5,6\rangle\cdot\langle 2,1,4\rangle\right)$$

$$= k300\left(2.4 + 2 + 1.2\right) + k12\left(8 + 5 + 24\right) = k(300\cdot 5.6 + 12\cdot 37) = 2124k$$