Name_____ ID_____

MATH 251   Exam 2   Spring 2007

Sections 509   Solutions   P. Yasskin

1-8	/40	11	/15
9	/ 5	12	/15
10	/15	13	/15
Total			/105

Multiple Choice: (5 points each. No part credit.)

1. Find the equation of the line perpendicular to the graph of

$xyz - x^2 - y^2 - z^2 = -8$ at the point $(1,2,3)$.

Where does this line intersect the xz-plane?

a. $(9,0,11)$

b. $(9,0,-5)$ Correct Choice

c. $(-7,0,11)$

d. $(-7,0,-5)$

e. $(5,0,-1)$

Let $F = xyz - x^2 - y^2 - z^2$.   Then $\vec{\nabla}F = \langle yz - 2x, xz - 2y, xy - 2z\rangle$.

Then the normal at $P = (1,2,3)$ is $\vec{N} = \vec{\nabla}F\big|_P = \langle 4,-1,-4\rangle$.

This is also the tangent vector to the perpendicular line. So $\vec{v} = \langle 4,-1,-4\rangle$.

So the perpendicular line is $X = P + t\vec{v} = (1,2,3) + t\langle 4,-1,-4\rangle$ or

$x = 1 + 4t$ $y = 2 - t$ $z = 3 - 4t$

This intersects the xz-plane when $y = 0$ or $t = 2$.

So $x = 1 + 4t = 9$ and $z = 3 - 4t = -5$.

2. The point $(x,y) = (1,2)$ is a critical point of the function $f(x,y) = (x^2 + y^2 - 4)^2 - 4x - 8y$.
Use the Second Derivative Test to classify it as a

a. local minimum   Correct Choice
b. local maximum
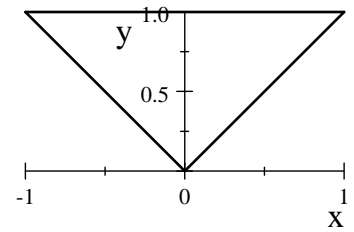**c**. inflection point
d. saddle point
e. Test Fails

$f_x = 2(x^2 + y^2 - 4)2x - 4$ $f_y = 2(x^2 + y^2 - 4)2y - 8$ $f_x(1,2) = f_y(1,2) = 0$
$f_{xx} = 2(2x)2x + 2(x^2 + y^2 - 4)2$ $f_{yy} = 2(2y)2y + 2(x^2 + y^2 - 4)2$ $f_{xy} = 2(2y)2x$
$f_{xx}(1,2) = 8 + 4 = 12 > 0$ $f_{yy}(1,2) = 32 + 4 = 36 > 0$ $f_{xy}(1,2) = 16$
$D = f_{xx}f_{yy} - f_{xy}^2 = 12 \cdot 36 - 16^2 = 176 > 0$ local minimum

1

3. Find the center of mass of the triangle
with vertices $(0,0)$, $(1,1)$ and $(-1,1)$
if the mass density is $\rho = y$.

a. $\left(0, \frac{4}{5}\right)$

b. $\left(0, \frac{3}{4}\right)$ Correct Choice

c. $\left(0, \frac{2}{3}\right)$

d. $\left(0, \frac{1}{2}\right)$
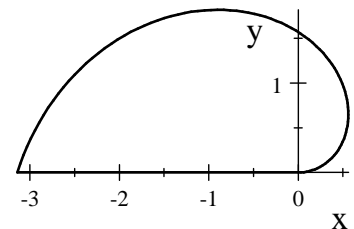
e. $\left(0, \frac{1}{3}\right)$

$\bar{x} = 0$ by symmetry,   Use a y-integral. $0 \le y \le 1$ $x = -y$ $x = y$

$$M = \iint \rho\, dA = \int_0^1 \int_{-y}^{y} y\, dx\, dy = \int_0^1 \left[yx \right]_{x=-y}^{y} dy = 2\int_0^1 y^2 dy = 2\left[\frac{y^3}{3} \right]_{y=0}^{1} = \frac{2}{3}$$

$$y\text{-mom} = M_x = \iint y\rho\, dA = \int_0^1 \int_{-y}^{y} y^2 dx\, dy = \int_0^1 \left[y^2 x \right]_{x=-y}^{y} dy = 2\int_0^1 y^3 dy = 2\left[\frac{y^4}{4} \right]_{y=0}^{1} = \frac{1}{2}$$

$$\bar{y} = \frac{y\text{-mom}}{M} = \frac{M_x}{M} = \frac{1}{2}\frac{3}{2} = \frac{3}{4}$$

4. Compute $\iint (x^2 + y^2)\, dA$ over the region bounded
by the polar curve $r = \theta$ and the x-axis.

a. $\frac{\pi^3}{6}$

b. $\frac{\pi^3}{9}$

c. $\frac{\pi^4}{12}$

d. $\frac{\pi^4}{16}$

e. $\frac{\pi^5}{20}$ Correct Choice

$$\int_0^\pi \int_0^\theta r^2\, r\, dr\, d\theta = \int_0^\pi \left[\frac{r^4}{4} \right]_0^\theta d\theta = \frac{1}{4}\int_0^\pi \theta^4\, d\theta = \frac{1}{4}\left[\frac{\theta^5}{5} \right]_0^\pi = \frac{\pi^5}{20}$$

2

5. Find the mass of the $1/8$ of the solid sphere $x^2 + y^2 + z^2 \le 16$ in the first octant if the mass density is $\delta = z$.

    a. π

    b. 4π

    c. 8π

    d. 16π Correct Choice

    e. 64π

$$M = \iiint \delta\,dV = \iiint z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho(\cos\varphi)\,\rho^2(\sin\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos\varphi\sin\varphi\,d\varphi \int_0^4 \rho^3\,d\rho = \frac{\pi}{2}\left[\frac{\sin^2\varphi}{2}\right]_{\varphi=0}^{\pi/2}\left[\frac{\rho^4}{4}\right]_{\rho=0}^4 = \frac{\pi}{2}\frac{1}{2}4^3 = 16\pi$$

6. Find the volume of the solid between the cone $z = 2\sqrt{x^2 + y^2}$ and the paraboloid $z = 8 - x^2 - y^2$. HINT: Find the radius where the cone and paraboloid intersect.

    a. $\dfrac{10\pi}{3}$

    b. $\dfrac{20\pi}{3}$

    c. $\dfrac{40\pi}{3}$ Correct Choice

    d. $\dfrac{80\pi}{3}$

    e. 30π

$$z = 2r = 8 - r^2 \qquad r^2 + 2r - 8 = 0 \qquad (r-2)(r+4) = 0 \qquad r = 2 \text{ since } r \ge 0.$$

$$V = \iiint 1\,dV = \int_0^{2\pi}\int_0^2\int_{2r}^{8-r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^2\left[rz\right]_{z=2r}^{8-r^2} dr = 2\pi\int_0^2 r(8 - r^2 - 2r)\,dr$$

$$= 2\pi\int_0^2(8r - r^3 - 2r^2)\,dr = 2\pi\left[4r^2 - \frac{r^4}{4} - \frac{2r^3}{3}\right]_{r=0}^2 = 2\pi\left(16 - 4 - \frac{16}{3}\right) = \frac{40\pi}{3}$$

7. Compute $\iint_S \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (z, z, x+y)$ over the surface S which is parametrized by $\vec{R}(u,v) = (u+v, u-v, uv)$ for $0 \le u \le 2$ and $0 \le v \le 3$ and oriented along $\vec{N} = \vec{e}_u \times \vec{e}_v$.

    a. 60

    b. 12 Correct Choice

    c. 0

    d. -12

    e. -60

$$\begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \end{array}$$

$$\vec{R}(u,v) = (u+v, u-v, uv) \qquad\qquad \vec{e}_u = (\;1,\quad 1,\quad v\;)$$

$$\vec{e}_v = (\;1,\quad -1,\quad u\;)$$

$$\vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(u+v) - \hat{\jmath}(u-v) + \hat{k}(-1-1) = (u+v, v-u, -2)$$

$$\vec{F} = (z, z, x+y) \qquad \vec{F}(\vec{R}(u,v)) = (uv, uv, 2u) \qquad \vec{F}\cdot\vec{N} = uv(u+v) + uv(v-u) + 2u(-2) = u(2v^2 - 4)$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint \vec{F}(\vec{R}(u,v))\cdot\vec{N}\,du\,dv = \int_0^3\int_0^2 u(2v^2 - 4)\,du\,dv = \left[\frac{u^2}{2}\right]_0^2\left[\frac{2v^3}{3} - 4v\right]_0^3 = (2)(18 - 12) = 12$$

8. Compute $\iint_C \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (xz, yz, z^2)$ over the cylindrical surface $x^2 + y^2 = 9$ for $0 \le z \le 2$ oriented outward.

    a. 36π Correct Choice

    b. $\dfrac{124}{3}\pi$

    c. 21π

    d. $\dfrac{62}{3}\pi$

    e. 18π

$$
\begin{array}{ccc}
\hat{\imath} & \hat{\jmath} & \hat{k}
\end{array}
$$

$$\vec{R}(\theta, z) = (3\cos\theta, 3\sin\theta, z) \qquad \vec{e}_\theta = \begin{pmatrix} -3\sin\theta, & 3\cos\theta, & 0 \end{pmatrix}$$

$$\vec{e}_z = \begin{pmatrix} 0, & 0, & 1 \end{pmatrix}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(3\cos\theta) - \hat{\jmath}(-3\sin\theta) + \hat{k}(0) = (3\cos\theta, 3\sin\theta, 0)$$

$$\vec{F} = (xz, yz, z^2) \qquad \vec{F}\big(\vec{R}(\theta, z)\big) = (3z\cos\theta, 3z\sin\theta, z^2) \qquad \vec{F} \cdot \vec{N} = 9z\cos^2\theta + 9z\sin^2\theta = 9z$$

$$\iint_C \vec{F} \cdot d\vec{S} = \iint \vec{F}\big(\vec{R}(\theta, z)\big) \cdot \vec{N} \, d\theta \, dz = \int_0^2 \int_0^{2\pi} 9z \, d\theta \, dz = 9(2\pi)\left[\frac{z^2}{2}\right]_0^2 = 36\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (5 points) At the right is the contour plot of a function $f(x, y)$. If you **start** at the dot at $(5, 6)$ and move so that your velocity is always in the direction of $\vec{\nabla}f,$ the gradient of $f,$ roughly sketch your path on the plot.

NOTE : The numbers on the right are the values of f on each level curve.
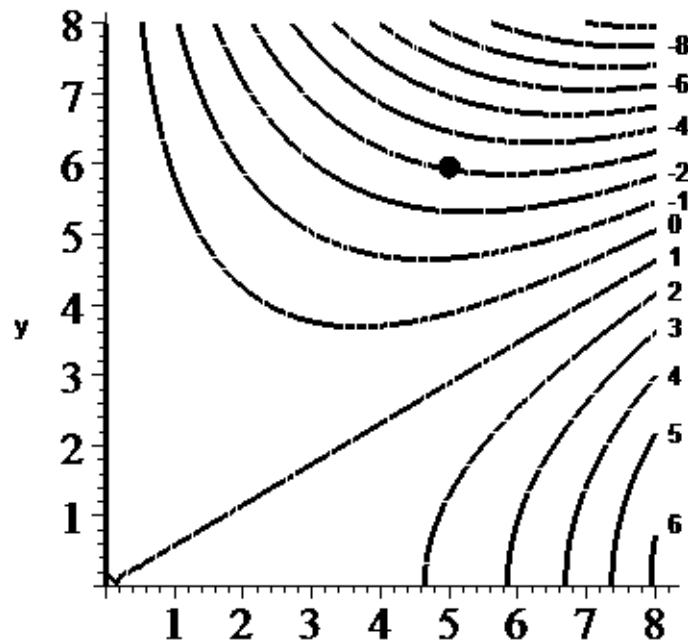
   The curve starts at $(5, 6)$ goes down and curves to the right towards higher values of the function $f,$ always perpendicular to each level curve. It should not go up.

10. (15 points) An aquarium in the shape of a rectangular solid has a base made of marble which costs 6 cents per square inch, a back made of mirrored glass which costs 2 cents per square inch and a front and sides made of clear glass which costs 1 cent per square inch. There is no top. If the volume of the aquarium is 4500 cubic inches, what are the dimensions of the cheapest such aquarium?

Let x be the length side to side, y be the width front to back, and z be the height.

The cost is $\quad C = 6xy + 2xz + 1(xz + 2yz) = 6xy + 3xz + 2yz$

The volume constraint is $\quad V = xyz = 4500 \quad$ which we solve for $\quad z = \dfrac{4500}{xy}$

So the cost becomes $\quad C = 6xy + \dfrac{13500}{y} + \dfrac{9000}{x}$

We want to minimize C. So we set the partials equal to zero:

$$C_x = 6y - \frac{9000}{x^2} = 0 \qquad\qquad C_y = 6x - \frac{13500}{y^2} = 0$$

$$y = \frac{1500}{x^2} \qquad x = \frac{2250}{y^2} = \frac{2250}{\left(\dfrac{1500}{x^2}\right)^2} = \frac{2250x^4}{1500 \cdot 1500} = \frac{1}{1000}x^4$$

Cancel an x and solve for x then y and z:

$$1 = \frac{x^3}{1000}, \qquad x = 10 \qquad y = \frac{1500}{x^2} = \frac{1500}{100} = 15 \qquad z = \frac{4500}{xy} = \frac{4500}{150} = 30$$

So the dimensions are: $\quad\boxed{x = 10 \qquad y = 15 \qquad z = 30}$
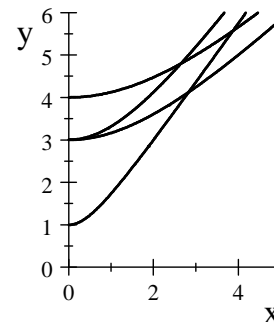
11. (15 points) Compute $\iint xy\,dA$ over the "diamond"

shaped region bounded by the curves

$$y^2 - x^2 = 9 \qquad y^2 - x^2 = 16$$

$$y^2 - 2x^2 = 1 \qquad y^2 - 2x^2 = 9$$

HINT: Let $u = y^2 - x^2$ and $v = y^2 - 2x^2$.

$$u - v = y^2 - x^2 - y^2 + 2x^2 = x^2 \qquad 2u - v = 2y^2 - 2x^2 - y^2 + 2x^2 = y^2 \qquad x = \sqrt{u - v} \qquad y = \sqrt{2u - v}$$

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left\| \begin{matrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\[2mm] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{matrix} \right\| = \left\| \begin{matrix} \dfrac{1}{2\sqrt{u-v}} & \dfrac{2}{2\sqrt{2u-v}} \\[2mm] \dfrac{-1}{2\sqrt{u-v}} & \dfrac{-1}{2\sqrt{2u-v}} \end{matrix} \right\|$$

$$= \left| \frac{1}{2\sqrt{u-v}}\frac{-1}{2\sqrt{2u-v}} - \frac{2}{2\sqrt{2u-v}}\frac{-1}{2\sqrt{u-v}} \right| = \frac{1}{4\sqrt{u-v}\sqrt{2u-v}}$$

$$xy = \sqrt{u-v}\sqrt{2u-v} \qquad\qquad 9 \le u \le 16 \qquad\qquad 1 \le v \le 9$$

$$\iint xy\,dA = \int_1^9 \int_9^{16} \sqrt{u-v}\sqrt{2u-v}\,\frac{1}{4\sqrt{u-v}\sqrt{2u-v}}\,du\,dv = \int_1^9 \int_9^{16} \frac{1}{4}\,du\,dv = \frac{1}{4}(16-9)(9-1) = 14$$

5

12. (15 points)   Find the average temperature on the hemisphere surface $x^2 + y^2 + z^2 = 4$,   with $z \geq 0$, if the temperature is $T = z$.

NOTE : The average of a function f is $f_{ave} = \dfrac{\iint f\, dS}{\iint dS}$.   HINT: Parametrize the hemisphere.

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \hat{i} \qquad\qquad \hat{j} \qquad\quad \hat{k}$$

$\vec{R}(\varphi,\theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$ $\qquad$ $\vec{e}_\varphi = (2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vec{e}_\theta = (-2\sin\varphi\sin\theta, 2\sin\varphi\cos\theta, \quad 0 \quad)$

$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = \hat{i}(4\sin^2\varphi\cos\theta) - \hat{j}(-4\sin^2\varphi\sin\theta) + \hat{k}(4\sin\varphi\cos\varphi\cos^2\theta + 4\sin\varphi\cos\varphi\sin^2\theta)$

$\qquad = (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$

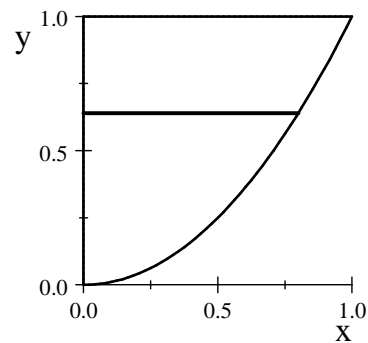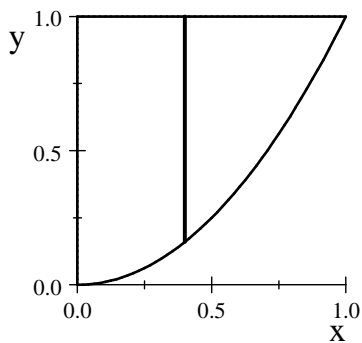$\left|\vec{N}\right| = \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi} = 4\sin\varphi$

$\iint dS = \int_0^{2\pi}\int_0^{\pi/2} \left|\vec{N}\right| d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 4\sin\varphi\, d\varphi\, d\theta = 8\pi$ $\qquad$ (Area of hemisphere $= \frac{1}{2}(4\pi R^2)$.)

$\iint T\, dS = \int_0^{2\pi}\int_0^{\pi/2} z\left|\vec{N}\right| d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\varphi\, 4\sin\varphi\, d\varphi\, d\theta = 8 \cdot 2\pi\left[\dfrac{\sin^2\varphi}{2}\right]_{\varphi=0}^{\pi/2} = 8\pi$

$T_{ave} = \dfrac{\iint T\, dS}{\iint dS} = \dfrac{8\pi}{8\pi} = 1$

13. (15 points)   Sketch the region of integration and then compute the integral $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$.

Reverse the order of integration:

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy = \int_0^1 \left[\frac{x^4}{4}\sin(y^3)\right]_{x=0}^{\sqrt{y}} dy = \int_0^1 \frac{y^2}{4}\sin(y^3)\, dy$$

Substitute: $\quad u = y^3 \quad\quad du = 3y^2\, dy \quad\quad y^2\, dy = \frac{1}{3} du$

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx = \frac{1}{12}\int \sin(u)\, du = \frac{-1}{12}\cos(u) = \frac{-1}{12}\cos(y^3)\Big|_0^1$$

$$= \frac{-1}{12}[\cos 1 - \cos 0] = \frac{1}{12}(1 - \cos 1)$$