Name________________________ ID__________________

MATH 251 Final Exam Spring 2007

Sections 509 Solutions P. Yasskin

1-10	/60	12	/15
11	/20	13	/15
Total			/110

Multiple Choice: (6 points each. No part credit.)

1. Consider the triangle with vertices $A = (2,4)$, $B = (1,1)$ and $C = (0,3)$. Find the angle at B.

a. $30°$

b. $45°$ Correct Choice

c. $60°$

d. $120°$

e. $135°$

$\overrightarrow{BA} = A - B = (1,3) \qquad \overrightarrow{BC} = C - B = (-1,2)$

$\left|\overrightarrow{BA}\right| = \sqrt{1^2 + 3^2} = \sqrt{10} \qquad \left|\overrightarrow{BC}\right| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad \overrightarrow{BA} \cdot \overrightarrow{BC} = -1 + 6 = 5$

$\cos\theta = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = \dfrac{5}{\sqrt{5}\sqrt{10}} = \dfrac{1}{\sqrt{2}} \qquad \theta = 45°$

2. For the "twisted cubic" **curve** $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ find the tangential acceleration $a_T = \hat{T} \cdot \vec{a}$.

a. $4t + 8t^2$

b. $\dfrac{1}{4t + 8t^2}$

c. $4t$ Correct Choice

d. $\dfrac{4t}{1 + 2t^2}$

e. $\dfrac{1}{1 + 2t^2}$

$\vec{v} = (1, 2t, 2t^2) \qquad |\vec{v}| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2 \qquad \hat{T} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{1 + 2t^2}(1, 2t, 2t^2)$

$\vec{a} = (0, 2, 4t) \qquad a_T = \hat{T} \cdot \vec{a} = \dfrac{1}{1 + 2t^2}(1, 2t, 2t^2) \cdot (0, 2, 4t) = \dfrac{1}{1 + 2t^2}(4t + 8t^2) = 4t$

OR $\quad a_T = \dfrac{d}{dt}|\vec{v}| = \dfrac{d}{dt}(1 + 2t^2) = 4t$

3. Find the linear approximation to $f(x,y) = (x^2+4)(y^3+1)$ at $(x,y) = (2,1)$.
Use it to estimate $f(2.1, 1.1)$.

a. 19.6

b. 19.2 Correct Choice

c. 16.2

d. 12.8

e. 5.87

$f(x,y) = (x^2+4)(y^3+1) \qquad f(2,1) = 16$
$f_x(x,y) = 2x(y^3+1) \qquad f_x(2,1) = 8$
$f_y(x,y) = (x^2+4)3y^2 \qquad f_y(2,1) = 24$
$f_{\text{tan}}(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 16 + 8(x-2) + 24(y-1)$
$f_{\text{tan}}(2.1,1.1) = 16 + 8(2.1-2) + 24(1.1-1) = 16 + .8 + 2.4 = 19.2$

4. Find the equation of the plane tangent to the surface $3x\sin z + y\cos z = 4$ at $(x,y,z) = \left(\sqrt{2}, \sqrt{2}, \frac{\pi}{4}\right)$.
What is the z-intercept?

a. $\frac{\pi}{2}$

b. $8+\pi$

c. $4+\frac{\pi}{2}$

d. $2+\frac{\pi}{4}$ Correct Choice

e. $1+\frac{\pi}{8}$

$f(x,y,z) = 3x\sin z + y\cos z \qquad P = \left(\sqrt{2}, \sqrt{2}, \frac{\pi}{4}\right)$
$\vec{\nabla} f = (3\sin z, \cos z, 3x\cos z - y\sin z) \qquad \vec{N} = \vec{\nabla} f\big|_P = \left(\frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2\right)$
$\vec{N} \cdot X = \vec{N} \cdot P \qquad \frac{3}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + 2z = \frac{3}{\sqrt{2}}\sqrt{2} + \frac{1}{\sqrt{2}}\sqrt{2} + 2\frac{\pi}{4} = 4 + \frac{\pi}{2}$
z-intercept: $x = 0 \qquad y = 0 \qquad 2z = 4 + \frac{\pi}{2} \qquad z = 2 + \frac{\pi}{4}$

5. Find the arc length of the curve $\vec{r}(t) = (2t^2, t^3)$ between $t = 0$ and $t = 1$.

a. $\dfrac{31}{27}$

b. $\dfrac{61}{27}$ Correct Choice

c. $\dfrac{91}{27}$

d. $\dfrac{31}{9}$

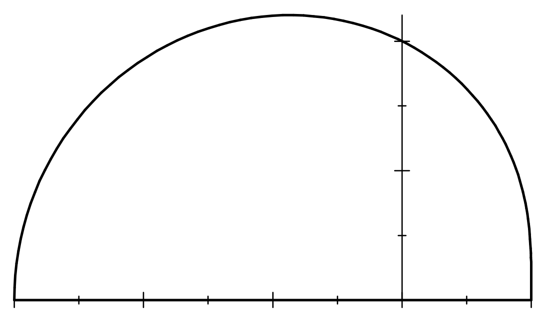
e. $\dfrac{61}{9}$

$\vec{v} = (4t, 3t^2) \qquad |\vec{v}| = \sqrt{16t^2 + 9t^4} = t\sqrt{16 + 9t^2}$

$$L = \int ds = \int |\vec{v}|\,dt = \int_0^1 t\sqrt{16 + 9t^2}\,dt = \frac{1}{27}\left[(16 + 9t^2)^{3/2}\right]_0^1 = \frac{1}{27}(25^{3/2} - 16^{3/2})$$

$$= \frac{1}{27}(125 - 64) = \frac{61}{27}$$

6. Find the mass of the region inside the upper half of the limaçon $r = 2 - \cos\theta$ if the surface density is $\rho = y$.



a. $\dfrac{5}{3}$

b. $\dfrac{10}{3}$

c. $\dfrac{13}{3}$

d. $\dfrac{15}{3}$

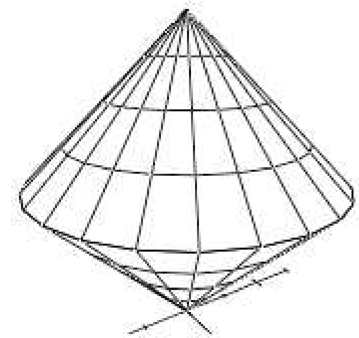
e. $\dfrac{20}{3}$ Correct Choice

$$M = \iint \rho\,dA = \iint y\,dA = \int_0^\pi \int_0^{2-\cos\theta} r\sin\theta\, r\,dr\,d\theta = \int_0^\pi \left[\frac{r^3}{3}\right]_{r=0}^{2-\cos\theta} \sin\theta\,d\theta = \frac{1}{3}\int_0^\pi (2 - \cos\theta)^3 \sin\theta\,d\theta$$

$u = 2 - \cos\theta \qquad du = \sin\theta\,d\theta$

$$M = \frac{1}{3}\int_1^3 u^3\,du = \left[\frac{u^4}{12}\right]_1^3 = \frac{81 - 1}{12} = \frac{20}{3}$$

7. Find the mass of the solid between the cones $z = r$ and $z = 6 - r$ if the volume mass density is $\rho = z$.



a. 54π Correct Choice

b. 108π

c. 216π

d. $\dfrac{297}{8}\pi$

e. $\dfrac{297}{4}\pi$

$$M = \iiint \rho\, dV = \iint z\, dV = \int_0^{2\pi}\int_0^3\int_r^{6-r} z\, r\, dz\, dr\, d\theta = 2\pi\int_0^3\left[\frac{z^2}{2}\right]_{z=r}^{6-r} r\, dr = \pi\int_0^3\left[(6-r)^2 - r^2\right] r\, dr$$

$$= \pi\int_0^3 [36 - 12r]\, r\, dr = \pi\int_0^3 (36r - 12r^2)\, dr = \pi[18r^2 - 4r^3]_0^3 = \pi(162 - 108) = 54\pi$$

8. Compute $\int_{\vec r} \vec F \cdot d\vec s$ for $\vec F = (2x + y + z, 2y + x + z, 2z + x + y)$ along the curve $\vec r(t) = (t\cos t, t\sin t, te^{t/\pi})$ between $t = 0$ and $t = \pi$.

HINT: Find a scalar potential and use the Fundamental Theorem of Calculus for Curves.

a. $\pi^2(1 + e^2 - e)$ Correct Choice

b. $\pi^2(1 + e^2 - 2e)$

c. $\pi^2(1 + e^2 + e)$

d. $\pi^2(1 + e^2 + 2e)$

e. $\pi^2(1 + e^2)$

$\vec F = \vec\nabla f$ for $f = x^2 + y^2 + z^2 + xy + xz + yz$

$A = \vec r(0) = (0,0,0)$ $\quad B = \vec r(\pi) = (-\pi, 0, \pi e)$

$$\int_{\vec r} \vec F \cdot d\vec s = \int_{\vec r} \vec\nabla f \cdot d\vec s = f(B) - f(A) = \pi^2 + \pi^2 e^2 - \pi^2 e$$

9. Compute $\oint_C \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-x^2y + x^3 - y^3, xy^2 + x^3 - y^3)$ counterclockwise around the circle $x^2 + y^2 = 9$.

HINT: Use Green's Theorem.

a. 36π

b. 72π

c. 144π

d. 162π Correct Choice

e. 324π

$P = -x^2y + x^3 - y^3 \qquad Q = xy^2 + x^3 - y^3 \qquad \frac{\partial Q}{\partial x} = y^2 + 3x^2 \qquad \frac{\partial P}{\partial y} = -x^2 - 3y^2$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4x^2 + 4y^2 = 4r^2$

$\oint_C \vec{F} \cdot d\vec{s} = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_0^{2\pi} \int_0^3 4r^2\, r\, dr\, d\theta = 2\pi[r^4]_0^3 = 162\pi$

10. Compute $\iint_{\partial C} \vec{F} \cdot d\vec{S}$ over the complete boundary of the cylinder $x^2 + y^2 \leq 4$ for $0 \leq z \leq 5$

for the vector field $\vec{F} = (x^3 + xy^2, x^2y + y^3, x^2z + y^2z)$.

HINT: Use Gauss' Theorem.

a. $\frac{80}{3}\pi$

b. 40π

c. 100π

d. $\frac{400}{3}\pi$

e. 200π Correct Choice

$\nabla \cdot \vec{F} = (3x^2 + y^2) + (x^2 + 3y^2) + (x^2 + y^2) = 5x^2 + 5y^2$

$\iint_{\partial C} \vec{F} \cdot d\vec{S} = \iiint \nabla \cdot \vec{F}\, dV = \int_0^5 \int_0^{2\pi} \int_0^2 (5r^2)\, r\, dr\, d\theta\, dz = 25(2\pi)\left[\frac{r^4}{4}\right]_0^2 = 200\pi$

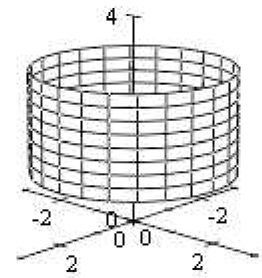
Work Out: (Points indicated. Part credit possible. Show all work.)

11. (20 points) Verify Stokes' Theorem $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$

for the vector field $\vec{F} = (yz^2, -xz^2, z^3)$ and the cylinder $x^2 + y^2 = 9$ for $1 \le z \le 2$ oriented out.

Be sure to check and explain the orientations.

Use the following steps:



a. The cylindrical surface may be parametrized by $\vec{R}(\theta, z) = (3\cos\theta, 3\sin\theta, z)$.
Compute the surface integral:
Successively find: $\vec{e}_\theta$, $\vec{e}_z$, $\vec{N}$, check orientation, $\vec{\nabla} \times \vec{F}$, $\vec{\nabla} \times \vec{F}\big(\vec{R}(\theta, z)\big)$, $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$

$$\begin{array}{rccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta = & (-3\sin\theta, & 3\cos\theta, & 0) \\ \vec{e}_z = & (\quad 0, & 0, & 1) \end{array}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(3\cos\theta) - \hat{\jmath}(-3\sin\theta) + \hat{k}(0) = (3\cos\theta, 3\sin\theta, 0)$$

$\vec{N}$ has the correct orientation.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & -xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0 - -2xz) - \hat{\jmath}(0 - 2yz) + \hat{k}(-z^2 - z^2) = (2xz, 2yz, -2z^2)$$

$$\vec{\nabla} \times \vec{F}\big(\vec{R}(\theta, z)\big) = (6z\cos\theta, 6z\sin\theta, -2z^2)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = 18z\cos^2\theta + 18z\sin^2\theta = 18z$$

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_C \vec{\nabla} \times \vec{F} \cdot \vec{N}\, d\theta\, dz = \int_1^2 \int_0^{2\pi} 18z\, d\theta\, dz = 2\pi[9z^2]_{z=1}^2 = 54\pi$$

b. Let U be the upper circle. Parametrize U and compute the line integral.

Successively find: $\vec{r}(\theta)$, $\vec{v}(\theta)$, check orientation, $\vec{F}(\vec{r}(\theta))$, $\oint_U \vec{F} \cdot d\vec{s}$.

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 2)$

$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$

By the right hand rule the upper curve must be traversed clockwise but $\vec{v}$ points counterclockwise. So reverse $\vec{v}$:

$\vec{v}(\theta) = (3\sin\theta, -3\cos\theta, 0)$

$\vec{F}(\vec{r}(\theta)) = (yz^2, -xz^2, z^3) = (12\sin\theta, -12\cos\theta, 8)$

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 36\sin^2\theta + 36\cos^2\theta\, d\theta = \int_0^{2\pi} 36\, d\theta = 72\pi$$

c. Let L be the lower circle. Parametrize L and compute the line integral.

Successively find: $\vec{r}(\theta)$, $\vec{v}(\theta)$, check orientation, $\vec{F}(\vec{r}(\theta))$, $\oint_L \vec{F} \cdot d\vec{s}$.

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 1)$

$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$

By the right hand rule the lower curve must be traversed counterclockwise and $\vec{v}$ is counterclockwise.

$\vec{F}(\vec{r}(\theta)) = (yz^2, -xz^2, z^3) = (3\sin\theta, -3\cos\theta, 1)$

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -9\sin^2\theta - 9\cos^2\theta\, d\theta = -\int_0^{2\pi} 9\, d\theta = -18\pi$$

d. Combine $\oint_U \vec{F} \cdot d\vec{s}$ and $\oint_L \vec{F} \cdot d\vec{s}$ to get $\oint_{\partial C} \vec{F} \cdot d\vec{s}$.

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \oint_U \vec{F} \cdot d\vec{s} + \oint_L \vec{F} \cdot d\vec{s} = 72\pi - 18\pi = 54\pi$$

which agrees with part (a).

12. (15 points) Find 3 numbers a, b and c whose sum is 12 for which $ab + 2ac + 3bc$ is a maximum.

Maximize $f = ab + 2ac + 3bc$ subject to the constraint $a + b + c = 12$.

Solve the constraint: $c = 12 - a - b$ Substitute into the function:

$f = ab + 2a(12 - a - b) + 3b(12 - a - b) = 24a + 36b - 2a^2 - 4ab - 3b^2$

Set the partial derivatives equal to zero and solve:

$f_a = 24 - 4a - 4b = 0 \qquad 4a + 4b = 24$
$f_b = 36 - 4a - 6b = 0 \qquad 4a + 6b = 36$

$2b = 12 \qquad b = 6 \qquad a + b = 6 \qquad a = 6 - b = 0$

$c = 12 - a - b = 12 - 0 - 6 = 6 \qquad \boxed{a = 0, \quad b = 6, \quad c = 6}$

13. (15 points) Find the mass and center of mass of the conical **surface** $z = \sqrt{x^2 + y^2}$ for $z \le 2$ with density $\rho = x^2 + y^2$. The cone may be parametrize as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.

$$\begin{array}{rccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 1) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$

$\left|\vec{N}\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{2} \qquad \rho = r^2$

$M = \iint \rho\, dS = \int\int r^2 \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^2 r^3\sqrt{2}\, dr\, d\theta = 2\pi\sqrt{2}\left[\frac{r^4}{4}\right]_0^2 = 8\pi\sqrt{2}$

$\bar{x} = \bar{y} = 0$ by symmetry.

$z\text{-mom} = M_{xy} = \iint z\rho\, dS = \int\int r^3\left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^2 r^4\sqrt{2}\, dr\, d\theta = 2\pi\sqrt{2}\left[\frac{r^5}{5}\right]_0^2 = \frac{64\pi\sqrt{2}}{5}$

$\bar{z} = \frac{z\text{-mom}}{M} = \frac{M_{xy}}{M} = \frac{64\pi\sqrt{2}}{5}\frac{1}{8\pi\sqrt{2}} = \frac{8}{5}$