Name______________________ Sec________

1-15	/60	17	/15
16	/15	18	/15
Total			/105

MATH 251/253 Exam 1 Ver. A Spring 2008

Sections 508,200,501,502 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. The triangle with vertices $A = (4,1,5)$, $B = (2,3,4)$ and $C = (3,5,2)$ is

a. scalene

b. isosceles and right

c. right but not isosceles

d. isosceles but not right Correct Choice

e. equilateral

$|AB| = \sqrt{2^2 + 2^2 + 1^2} = 3$ $|BC| = \sqrt{1^2 + 2^2 + 2^2} = 3$ $|AC| = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}$

2. Find the area of the triangle with vertices $A = (4,1,5)$, $B = (2,3,4)$ and $C = (3,5,2)$.

a. $\sqrt{65}$

b. $\frac{1}{2}\sqrt{65}$ Correct Choice

c. 130

d. 65

e. $\frac{65}{2}$

$\overrightarrow{AB} = (-2,2,-1)$ $\overrightarrow{AC} = (-1,4,-3)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 2 & -1 \\ -1 & 4 & -3 \end{vmatrix} = \hat{\imath}(-6+4) - \hat{\jmath}(6-1) + \hat{k}(-8+2) = (-2,-5,-6)$$

$$A_{tri} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{4+25+36} = \frac{1}{2}\sqrt{65}$$

3. Find an equation of the plane containing the points $A = (4,1,5)$, $B = (2,3,4)$ and $C = (3,5,2)$.

a. $2x - 5y - 6z = -27$

b. $2x + 5y - 6z = -17$

c. $2x + 5y + 6z = 43$ Correct Choice

d. $2x - 5y + 6z = 33$

e. $2x - 5y + 6z = 13$

$\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC} = (-2,-5,-6)$ $\quad \vec{N} \cdot X = \vec{N} \cdot A$

$-2x - 5y - 6z = -2(4) - 5(1) - 6(5) = -43$

4. Find the point where the line $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{4}$ intersects the plane $x + y - z = 2$.
Then $x + y + z =$

a. 18 Correct Choice

b. 28

c. 14

d. 22

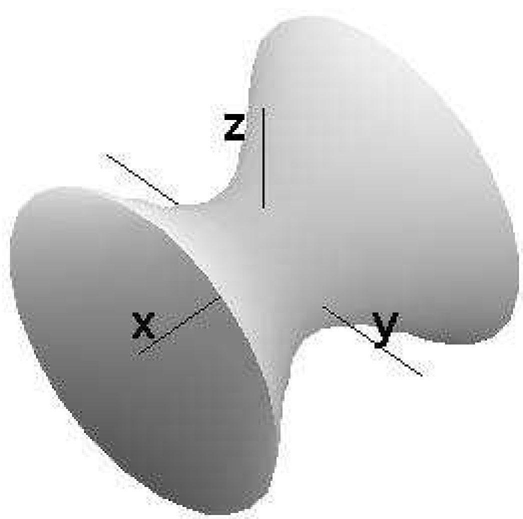
e. 2

The parametric equation for the line is $(x,y,z) = (2+2t, 3+3t, 4+4t)$.

Substitute into the plane: $(2+2t) + (3+3t) - (4+4t) = 2$ or $1 + t = 2$.

So $t = 1$ and $(x,y,z) = (4,6,8)$. So $x + y + z = 18$.

5. Which of the following is the equation of the surface?



a. $x^2 - y^2 - z^2 = 1$

b. $x^2 - y^2 - z^2 = 0$

c. $x^2 - y^2 - z^2 = -1$ Correct Choice

d. $x - y^2 - z^2 = 0$

e. $x + y^2 + z^2 = 0$

The plot is a hyperboloid. (a) and (c) are hyperboloids. (b) is a cone. (d) and (e) are paraboloids.

(a) is $x^2 = 1 + y^2 + z^2$. So $x^2 \geq 1$, which is false here.

(c) is $y^2 + z^2 = 1 + x^2$. So y and z stay outside a circle of radius 1, which is true here.

6. Find the line tangent to the curve $\vec{r}(t) = (3t, -3t^2, 2t^3)$ at the point $(3, -3, 2)$.

a. $(x, y, z) = (3 + 3t, -3 - 3t^2, 2 + 2t^3)$

b. $(x, y, z) = (3 + 3t, -3 + 6t^2, 2 + 6t^3)$

c. $(x, y, z) = (3 + 3t, -3 - 6t^2, 2 + 6t^3)$

d. $(x, y, z) = (3 + 3t, -3 + 6t, 2 + 6t)$

e. $(x, y, z) = (3 + 3t, -3 - 6t, 2 + 6t)$ Correct Choice

$(3, -3, 2) = \vec{r}(1) \qquad \vec{v}(t) = (3, -6t, 6t^2) \qquad \vec{v}(1) = (3, -6, 6)$

$(x, y, z) = \vec{r}(1) + t\vec{v}(1) = (3, -3, 2) + t(3, -6, 6) = (3 + 3t, -3 - 6t, 2 + 6t)$

7. Find the arc length of the curve $\vec{r}(t) = (3t, -3t^2, 2t^3)$ between $(0, 0, 0)$ and $(3, -3, 2)$.

a. 5 Correct Choice

b. 4

c. 3

d. 2

e. 1

$\vec{v} = (3, -6t, 6t^2) \qquad |\vec{v}| = \sqrt{9 + 36t^2 + 36t^4} = 3\sqrt{1 + 4t^2 + 4t^4} = 3(1 + 2t^2)$

$L = \int_0^1 |\vec{v}|\, dt = \int_0^1 3(1 + 2t^2)\, dt = 3\left(t + \frac{2}{3}t^3\right) = \left[3t + 2t^3\right]_0^1 = 5$

8. Find the tangential acceleration of the curve $\vec{r}(t) = (3t, -3t^2, 2t^3)$.

a. $6t$

b. $12t$ Correct Choice

c. $36t$

d. $3t + 2t^3$

e. $3t - 2t^3$

$\vec{v} = (3, -6t, 6t^2) \qquad |\vec{v}| = \sqrt{9 + 36t^2 + 36t^4} = 3\sqrt{1 + 4t^2 + 4t^4} = 3 + 6t^2$

$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(3 + 6t^2) = 12t$

9. A jet fighter flies along the parabola $z = x^2$ in the xz-plane toward increasing values of x. Then, . . .

HINT: There are no computations.

a. $\hat{N} = (0,0,-1)$ at all times.

b. $\hat{N} = (0,0,1)$ at all times.

c. $\hat{N} = (1,0,1)$ at all times.

d. $\hat{B} = (0,-1,0)$ at all times. Correct Choice

e. $\hat{B} = (0,1,0)$ at all times.

$\hat{T}$ points tangent to the parabola toward increasing x.

$\hat{N}$ points perpendicular to the parabola toward the inside which is basically up.

So $\hat{B} = \hat{T} \times \hat{N}$ points toward the negative y-direction by the RHR. So $\hat{B} = (0,-1,0)$.

10. If $f(x,y) = x^2 e^{xy}$, which of the following is FALSE?

a. $f_x(1,2) = 4e^2$

b. $f_y(1,2) = e^2$

c. $f_{xx}(1,2) = 14e^2$

d. $f_{xy}(1,2) = 5e^2$

e. $f_{yy}(1,2) = 4e^2$ Correct Choice

$f_x = 2xe^{xy} + x^2 ye^{xy}$	$f_y = x^3 e^{xy}$	$f_{xx} = 2e^{xy} + 4xye^{xy} + x^2y^2e^{xy}$	$f_{xy} = 3x^2e^{xy} + x^3ye^{xy}$	$f_{yy} = x^4 e^{xy}$
$f_x(1,2) = 4e^2$	$f_y(1,2) = e^2$	$f_{xx}(1,2) = 14e^2$	$f_{xy}(1,2) = 5e^2$	$f_{yy}(1,2) = e^2$

11. Find the plane tangent to the graph of the function $f(x,y) = x^2y^3$ at $(x,y) = (1,2)$.
The z-intercept is

a. -40

b. -8

c. $+8$

d. -32 Correct Choice

e. $+32$

$f = x^2y^3$ $\quad f_x = 2xy^3$ $\quad f_y = 3x^2y^2$

$f(1,2) = 8$ $\quad f_x(1,2) = 16$ $\quad f_y(1,2) = 12$

$$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 8 + 16(x-1) + 12(y-2) = 16x + 12y - 32$$

The z-intercept is -32.

12. Find the unit vector direction in which the function $f(x,y) = x^2y^3$ increases most rapidly at the point $(x,y) = (1,2)$.

a. $\left(\frac{3}{5}, \frac{4}{5}\right)$

b. $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

c. $\left(\frac{4}{5}, \frac{3}{5}\right)$ Correct Choice

d. $\left(-\frac{4}{5}, -\frac{3}{5}\right)$

e. $\left(\frac{4}{5}, -\frac{3}{5}\right)$

$$\vec{\nabla} f = (2xy^3, 3x^2y^2) \qquad \vec{v} = \vec{\nabla} f(1,2) = (16,12) \qquad \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \left(\frac{4}{5}, \frac{3}{5}\right)$$

13. Find an equation of the plane tangent to the surface $x^2z + yz^3 = 11$ at the point $(x,y,z) = (3,2,1)$.

a. $6x + y + 15z = 35$ Correct Choice

b. $6x - y + 15z = 31$

c. $18x - 2y + 15z = 65$

d. $3x + 2y + z = 14$

e. $3x - 2y + z = 6$

$$F = x^2z + yz^3 \qquad \vec{\nabla} F = (2xz, z^3, x^2 + 3yz^2) \qquad \vec{N} = \vec{\nabla} F\Big|_{(3,2,1)} = (6,1,15) \qquad P = (3,2,1)$$

$$\vec{N} \cdot X = \vec{N} \cdot P \qquad 6x + y + 15z = 6(3) + (2) + 15(1) = 35$$

14. An arch has the shape of the semi-circle $x^2 + y^2 = 16$ for $y \geq 0$ and has linear mass density given by $\rho = 6 - y$ so it is less dense at the top. Find the total mass of the arch.

NOTE: The arch may be parametrized by $\vec{r}(t) = (4\cos t, 4\sin t)$.

a. $20\pi - 16$

b. $20\pi - 32$

c. 24π

d. $24\pi - 16$

e. $24\pi - 32$ Correct Choice

$\vec{v} = (-4\sin t, 4\cos t)$ $\quad |\vec{v}| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$ $\quad \rho(\vec{r}(t)) = 6 - 4\sin t$

$$M = \int \rho\, ds = \int \rho(\vec{r}(t))|\vec{v}|\, dt = \int_0^\pi (6 - 4\sin t)4\, dt = 4\Big[6t + 4\cos t\Big]_0^\pi$$

$$= 4(6\pi - 4) - 4(0 + 4) = 24\pi - 32$$

15. Find the center of mass of the arch of problem 14.

a. $\left(0, \dfrac{12 - 4\pi}{3\pi - 4}\right)$

b. $\left(0, \dfrac{24 - 4\pi}{3\pi - 4}\right)$ Correct Choice

c. $\left(0, \dfrac{12 - 4\pi}{3\pi - 2}\right)$

d. $\left(0, \dfrac{24 - 4\pi}{3\pi - 2}\right)$

e. $\left(0, \dfrac{12 - 4\pi}{5\pi - 4}\right)$

$\bar{x} = 0$ by symmetry.

$$M_x = \int y\rho\, ds = \int y\rho(\vec{r}(t))|\vec{v}|\, dt = \int_0^\pi 4\sin t(6 - 4\sin t)4\, dt = 16\int_0^\pi (6\sin t - 4\sin^2 t)\, dt$$

$$= 16\int_0^\pi [6\sin t - 2(1 - \cos 2t)]\, dt = 16\Big[-6\cos t - 2\Big(t - \frac{\sin 2t}{2}\Big)\Big]_0^\pi$$

$$= 16(6 - 2(\pi)) - 16(-6) = 192 - 32\pi$$

$$\bar{y} = \frac{M_x}{M} = \frac{192 - 32\pi}{24\pi - 32} = \frac{24 - 4\pi}{3\pi - 4} \approx 2.1$$

Work Out: (15 points each. Part credit possible. Show all work.)

16. An object moves around **2 loops** of the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$

from $(3,0,0)$ to $(3,0,16\pi)$ under the action of a force $\vec{F} = (-y,x,z)$.

Find the work done by the force.

$\vec{v} = (-3\sin t, 3\cos t, 4) \qquad \vec{F}(\vec{r}(t)) = (-3\sin t, 3\cos t, 4t)$

$$W = \int_0^{4\pi} \vec{F} \cdot d\vec{s} = \int_0^{4\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v}\, dt = \int_0^{4\pi} (9\sin^2 t + 9\cos^2 t + 16t)\, dt = \int_0^{4\pi} (9 + 16t)\, dt$$

$$= \left[9t + 8t^2 \right]_0^{4\pi} = 36\pi + 128\pi^2$$

17. A cardboard box has length $L = 40$ cm, width $W = 30$ cm and height $H = 20$ cm.

The cardboard is 0.2 cm thick on each side and 0.4 cm thick on the top and bottom.

Use differentials to estimate the volume of the cardboard used to make the box.

$V = LWH \qquad \Delta L = \Delta W = 2 \times 0.2 = 0.4 \qquad \Delta H = 2 \times 0.4 = 0.8$

$$\Delta V \approx \frac{\partial V}{\partial L}\Delta L + \frac{\partial V}{\partial W}\Delta W + \frac{\partial V}{\partial H}\Delta H = WH\Delta L + LH\Delta W + LW\Delta H$$

$$= 30 \times 20 \times 0.4 + 40 \times 20 \times 0.4 + 40 \times 30 \times 0.8 = 240 + 320 + 960 = 1520$$

18. In a particular ideal gas the pressure, P, the temperature, T, and density, ρ, are related by $P = 5\rho T$.

Currently, the temperature is $T = 300°$K and decreasing at $2°$K/hr while the density is $\rho = 2 \times 10^{-4}$ gm/cm^3 and increasing at 4×10^{-6} gm/cm^3/hr.

Find the current pressure. Is it increasing or decreasing and at what rate?

The pressure is $P = 5\rho T = 5(2 \times 10^{-4})(300) = 0.3$ atm. By the chain rule,

$$\frac{dP}{dt} = \frac{\partial P}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt} = 5T\frac{d\rho}{dt} + 5\rho\frac{dT}{dt} = 5(300)(4 \times 10^{-6}) + 5(2 \times 10^{-4})(-2)$$

$$= 6 \times 10^{-3} - 2 \times 10^{-3} = 4 \times 10^{-3} \quad \text{increasing}$$