Name________________________ Sec________

1-13	/65
14	/25
15	/15
Total	/105

MATH 251 Final Exam Spring 2008

Sections 508 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Which of the following lines lies in the plane: $2x - y - z = 0$?

a. $(x,y,z) = (1,2,3) + t(1,1,1)$

b. $(x,y,z) = (3,2,1) + t(1,1,1)$

c. $(x,y,z) = (2,1,3) + t(1,1,1)$ Correct Choice

d. $(x,y,z) = (3,1,2) + t(1,1,1)$

e. $(x,y,z) = (1,3,2) + t(1,1,1)$

Plug each line into the plane:

(a) $2x - y - z = 2 - 2 - 3 + 2t - t - t = -3 \neq 0$

(b) $2x - y - z = 6 - 2 - 1 + 2t - t - t = 3 \neq 0$

(c) $2x - y - z = 4 - 1 - 3 + 2t - t - t = 0$

(d) $2x - y - z = 6 - 1 - 2 + 2t - t - t = 3 \neq 0$

(e) $2x - y - z = 2 - 3 - 2 + 2t - t - t = -3 \neq 0$

2. Find the equation of the plane tangent to the graph of the function $f(x,y) = x^2y + xy^2$ at the point $(2,1)$. Then the z-intercept is

a. -12 Correct Choice

b. -6

c. 0

d. 6

e. 12

$f = x^2y + xy^2$ $\quad f(2,1) = 6$ $\quad z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f_x = 2xy + y^2$ $\quad f_x(2,1) = 5$ $\quad = 6 + 5(x-2) + 8(y-1)$

$f_y = x^2 + 2xy$ $\quad f_y(2,1) = 8$ $\quad = 5x + 8y - 12$

3. Find the arc length of the curve $\vec{r}(t) = (e^t, 2t, 2e^{-t})$ between $(1,0,2)$ and $(e,2,2e^{-1})$.

Hint: Look for a perfect square.

a. $e - 2e^{-1}$

b. $1 + e - 2e^{-1}$ Correct Choice

c. $e - 2e^{-1} - 1$

d. $e + 2e^{-1}$

e. $e + 2e^{-1} - 3$

$\vec{v} = (e^t, 2, -2e^{-t}) \qquad |\vec{v}| = \sqrt{e^{2t} + 4 + 4e^{-2t}} = e^t + 2e^{-t}$

$\vec{r}(t) = (1,0,2)$ at $t = 0 \qquad \vec{r}(t) = (e,2,2e^{-1})$ at $t = 1$

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 (e^t + 2e^{-t})\,dt = [e^t - 2e^{-t}]_0^1 = (e - 2e^{-1}) - (1 - 2) = 1 + e - 2e^{-1}$$

4. Find the tangential acceleration a_T of the curve $\vec{r}(t) = (e^t, 2t, 2e^{-t})$.

Hint: Which formula is easier?

a. $e^t + 2e^{-t}$

b. $e^t + 4e^{-t}$

c. $e^t - 2e^{-t}$ Correct Choice

d. $e^t - 4e^{-t}$

e. $e^{2t} - 4e^{-2t}$

$\vec{v} = (e^t, 2, -2e^{-t}) \qquad |\vec{v}| = \sqrt{e^{2t} + 4 + 4e^{-2t}} = e^t + 2e^{-t} \qquad a_T = \dfrac{d|\vec{v}|}{dt} = e^t - 2e^{-t}$

OR

$\hat{T} = \dfrac{1}{e^t + 2e^{-t}}(e^t, 2, -2e^{-t}) \qquad \vec{a} = (e^t, 0, 2e^{-t}) \qquad a_T = \hat{T} \cdot \vec{a} = \dfrac{1}{e^t + 2e^{-t}}(e^{2t} - 4e^{-2t}) = e^t - 2e^{-t}$

5. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.
If the radius r is currently 3 cm and increasing at 2 cm/sec
while the height h is currently 4 cm and decreasing at 1 cm/sec,
is the volume increasing or decreasing and at what rate?

a. decreasing at 19π cm$^3/\text{sec}$

b. decreasing at 13π cm$^3/\text{sec}$

c. neither increasing nor decreasing

d. increasing at 13π cm$^3/\text{sec}$ Correct Choice

e. increasing at 19π cm$^3/\text{sec}$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt} = \frac{2}{3}\pi(3)(4)(2) + \frac{1}{3}\pi(3)^2(-1) = 13\pi$$

This is positive and so increasing.

6. Which of the following is a local maximum of $f(x,y) = \sin(x)\sin(y)$?

a. $(0,0)$

b. $\left(\frac{\pi}{2},0\right)$

c. (π,π)

d. $\left(0,\frac{\pi}{2}\right)$

e. None of the above Correct Choice

$f_x(x,y) = \cos(x)\sin(y) \qquad f_y(x,y) = \sin(x)\cos(y)$

Since $f_x\left(0,\frac{\pi}{2}\right) = f_y\left(\frac{\pi}{2},0\right) = 1$, the points $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2},0\right)$ are not even critical points.

$f_{xx}(x,y) = -\sin(x)\sin(y) \qquad f_{yy}(x,y) = -\sin(x)\sin(y) \qquad f_{xy}(x,y) = \cos(x)\cos(y)$

Since $f_{xx}(0,0) = 0 \qquad f_{yy}(0,0) = 0 \qquad f_{xy}(0,0) = 1$

we have $D(0,0) = f_{xx}f_{yy} - f_{xy}^2 = -1$ and $(0,0)$ is a saddle.

Since $f_{xx}(\pi,\pi) = 0 \qquad f_{yy}(\pi,\pi) = 0 \qquad f_{xy}(\pi,\pi) = 1$

we have $D(\pi,\pi) = f_{xx}f_{yy} - f_{xy}^2 = -1$ and $\left(0,\frac{\pi}{2}\right)$ is a saddle.

7. Find the equation of the plane tangent to the surface $\frac{x^2}{z^2} + \frac{y^3}{z^3} = 17$ at the point $(3,2,1)$.

Then the z-intercept is

a. 0 Correct Choice

b. -42

c. 42

d. -18

e. 18

Let $f = \frac{x^2}{z^2} + \frac{y^3}{z^3}$ and $P = (3,2,1)$. Then $\vec{\nabla} f = \left(\frac{2x}{z^2}, \frac{3y^2}{z^3}, \frac{-2x^2}{z^3} + \frac{-3y^3}{z^4} \right)$ and

$\vec{N} = \vec{\nabla} f\Big|_P = (6,12,-42)$

$\vec{N} \cdot X = \vec{N} \cdot P \qquad 6x + 12y - 42z = 6 \cdot 3 + 12 \cdot 2 - 42 \cdot 1 = 0$ If $x = y = 0$, then $z = 0$.

8. Compute $\int \vec{F} \cdot d\vec{s}$ counterclockwise around the circle $x^2 + y^2 = 4$

for the vector field $\vec{F} = \left(-x^4 y + \frac{1}{3} x^2 y^3, x y^4 + x^3 y^2 \right)$.

Hint: Use Green's Theorem and factor the integrand.

a. $\frac{4\pi}{3}$

b. $\frac{8\pi}{3}$

c. $\frac{16\pi}{3}$

d. $\frac{32\pi}{3}$

e. $\frac{64\pi}{3}$ Correct Choice

$\int \vec{F} \cdot d\vec{s} = \int P\,dx + Q\,dy$ where $P = -x^4 y + \frac{1}{3} x^2 y^3$ and $Q = x y^4 + x^3 y^2$

$$\int \vec{F} \cdot d\vec{s} = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \iint (y^4 + 3x^2y^2) - (-x^4 + x^2y^2)\,dx\,dy = \iint (x^4 + y^4 + 2x^2y^2)\,dx\,dy$$

$$= \iint (x^2 + y^2)^2\,dx\,dy = \int_0^{2\pi} \int_0^2 r^4\, r\,dr\,d\theta = 2\pi \left[\frac{r^6}{6} \right]_9^2 = \frac{64\pi}{3}$$

9. Find the mass of the half cylinder $x^2 + y^2 \le 4$ for $0 \le z \le 10$ and $y \ge 0$ if the density is $\rho = x^2 + y^2$.

a. 10π

b. 20π

c. 40π Correct Choice

d. 80π

e. 160π

$$M = \iiint \rho\, dV = \int_0^{10}\int_0^{\pi}\int_0^{2} r^2\, r\, dr\, d\theta\, dz = (10)(\pi)\left[\frac{r^4}{4}\right]_0^2 = 40\pi$$

10. Find the center of mass of the half cylinder $x^2 + y^2 \le 4$ for $0 \le z \le 10$ and $y \ge 0$ if the density is $\rho = x^2 + y^2$.

a. $(0,1,5)$

b. $\left(0, \frac{8}{5\pi}, 5\right)$

c. $\left(0, \frac{5\pi}{8}, 5\right)$

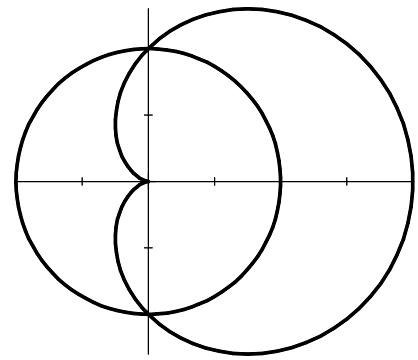
d. $\left(0, \frac{16}{5\pi}, 5\right)$ Correct Choice

e. $\left(0, \frac{5\pi}{16}, 5\right)$

$$M_{xz} = \iiint y\rho\, dV = \int_0^{10}\int_0^{\pi}\int_0^{2} r\sin\theta\, r^2\, r\, dr\, d\theta\, dz = (10)\left[-\cos\theta\right]_0^{\pi}\left[\frac{r^5}{5}\right]_0^2 = (10)(2)\left(\frac{32}{5}\right) = 128$$

$\bar{y} = \frac{M_{xz}}{M} = \frac{128}{40\pi} = \frac{16}{5\pi}$ By symmetry, $\bar{x} = 0$ $\bar{z} = 5$

11. Find the area inside the cardioid $r = 1 + \cos\theta$ but outside the circle $r = 1$.



a. $\frac{\pi}{4}$

b. $\frac{\pi}{2}$

c. $2 - \frac{\pi}{4}$

d. $2 + \frac{\pi}{4}$ Correct Choice

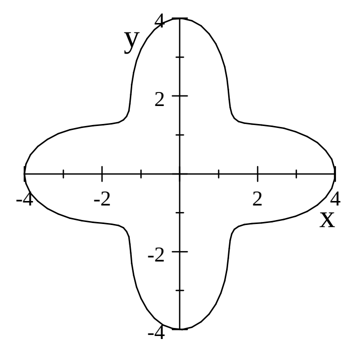
e. $2 - \frac{\pi}{2}$

$$A = \iint 1\,dA = \int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=1}^{1+\cos\theta} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left([1+\cos\theta]^2 - 1\right)d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}(2\cos\theta + \cos^2\theta)\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(2\cos\theta + \frac{1+\cos(2\theta)}{2}\right)d\theta$$

$$= \frac{1}{2}\left[2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\right]_{-\pi/2}^{\pi/2} = \frac{1}{2}\left[2(1) + \frac{1}{2}\left(\frac{\pi}{2}\right)\right] - \frac{1}{2}\left[2(-1) + \frac{1}{2}\left(\frac{-\pi}{2}\right)\right] = 2 + \frac{\pi}{4}$$

12. Compute $\oint \vec{\nabla} f \cdot d\vec{s}$ counterclockwise once around the polar curve $r = 3 + \cos(4\theta)$ for the function $f(x,y) = x^2 y$.



a. 2π

b. 4π

c. 6π

d. 8π

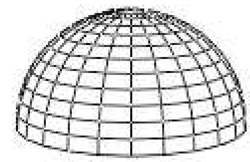
e. 0 Correct Choice

By the FTCC, $\int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A)$.

However, since it is a closed curve, $B = A$, and $\int_A^B \vec{\nabla} f \cdot d\vec{s} = 0$.

OR by Green's Theorem, $\oint \vec{\nabla} f \cdot d\vec{s} = \iint \vec{\nabla} \times \vec{\nabla} f \cdot \hat{k}\,dA = 0$ because $\vec{\nabla} \times \vec{\nabla} f = 0$ for any f.

13. Gauss' Theorem states $\iiint\limits_H \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial H} \vec{F} \cdot d\vec{S}$



Compute either integral for the solid hemisphere, H,

given by $x^2 + y^2 + z^2 \le 4$ with $z \ge 0$

and the vector field $\vec{F} = (xz^2, yz^2, 0)$.

Notice that the boundary of the solid hemisphere ∂H consists of the hemisphere surface S given by $x^2 + y^2 + z^2 = 4$ with $z \ge 0$ and the disk D given by $x^2 + y^2 \le 4$ with $z = 0$.

a. $\dfrac{64\pi}{15}$

b. $\dfrac{128\pi}{15}$ Correct Choice

c. $\dfrac{8}{3}\pi^2$

d. $\dfrac{32}{3}\pi^2$

e. $\dfrac{64}{3}\pi^2$

The volume integral: $\vec{\nabla} \cdot F = z^2 + z^2 + 0 = 2z^2 = 2\rho^2 \cos^2\varphi \qquad dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$

$$\iiint\limits_H \vec{\nabla} \cdot F\, dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^2 2\rho^2 \cos^2\varphi\, \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi = 2\pi\left[\frac{2\rho^5}{5}\right]_0^2 \left[\frac{-\cos^3\varphi}{3}\right]_0^{\pi/2}$$

$$= \frac{128\pi}{5}\left(\frac{-0}{3} - \frac{-1}{3}\right) = \frac{128\pi}{15}$$

The surface integral over the disk: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0) \qquad \vec{F}\left(\vec{R}(r,\theta)\right) = (0,0,0)$

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 0\, dr\, d\theta = 0$$

The surface integral over the hemisphere: $\vec{R}(\theta,\varphi) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta = & (-2\sin\varphi\sin\theta, & 2\sin\varphi\cos\theta, & 0\ ) \\ \vec{e}_\varphi = & (2\cos\varphi\cos\theta, & 2\cos\varphi\sin\theta, & -2\sin\varphi) \end{vmatrix}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_\varphi = \hat{\imath}(-4\sin^2\varphi\cos\theta) - \hat{\jmath}(4\sin^2\varphi\sin\theta)$$
$$+ \hat{k}(-4\sin\varphi\cos\varphi\sin^2\theta - 4\sin\varphi\cos\varphi\cos^2\theta)$$
$$= (-4\sin^2\varphi\cos\theta, -4\sin^2\varphi\sin\theta, -4\sin\varphi\cos\varphi)$$

Reverse $\vec{N} = (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$

$\vec{F}\left(\vec{R}(r,\theta)\right) = (8\sin\varphi\cos^2\varphi\cos\theta, 8\sin\varphi\cos^2\varphi\sin\theta, 0)$

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_S \vec{F} \cdot \vec{N}\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 32\sin^3\varphi\cos^2\varphi\, d\varphi\, d\theta = 2\pi\int_0^{\pi/2} 32\sin\varphi(1-\cos^2\varphi)\cos^2\varphi\, d\varphi$$

Let $u = \cos\varphi \qquad du = -\sin\varphi\, d\varphi$

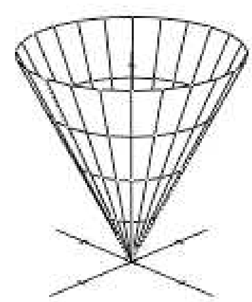
$$\iint\limits_S \vec{F} \cdot d\vec{S} = -2\pi\int_1^0 32(1-u^2)u^2\, du = -64\pi\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_1^0 = 64\pi\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{128\pi}{15}$$

Work Out: (Part credit possible. Show all work.)

14. (25 points) Verify Stokes' Theorem $\iint_C \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_{\partial C} \vec{F}\cdot d\vec{s}$

for the cone C given by $z = 2\sqrt{x^2+y^2}$ for $z \le 8$

oriented up and in, and the vector field $\vec{F} = (yz, -xz, z)$.



Be sure to check and explain the orientations. Use the following steps:

a. Note: The cone may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2r)$

Compute the surface integral by successively finding:

$$\vec{e}_r,\quad \vec{e}_\theta,\quad \vec{N},\quad \vec{\nabla}\times\vec{F},\quad \vec{\nabla}\times\vec{F}\big(\vec{R}(r,\theta)\big),\quad \iint_C \vec{\nabla}\times\vec{F}\cdot d\vec{S}$$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta & \sin\theta, & 2) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_r\times\vec{e}_\theta = \hat{\imath}(-2r\cos\theta) - \hat{\jmath}(2r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-2r\cos\theta, -2r\sin\theta, r)$$

oriented correctly

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz, & -xz, & z \end{vmatrix} = \hat{\imath}(0 - -x) - \hat{\jmath}(0 - y) + \hat{k}(-z - z) = (x, y, -2z)$$

$$\vec{\nabla}\times\vec{F}\big(\vec{R}(r,\theta)\big) = (r\cos\theta, r\sin\theta, -4r)$$

$$\iint_C \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \iint_C \vec{\nabla}\times\vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^4 (-2r^2\cos^2\theta - 2r^2\sin^2\theta - 4r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 (-6r^2)\,dr\,d\theta = 2\pi[-2r^3]_0^4 = -256\pi$$

Recall: $\vec{F} = (yz, -xz, z)$

b. Compute the line integral by parametrizing the boundary curve and successively finding:

$$\vec{r}(\theta), \quad \vec{v}, \quad \vec{F}(\vec{r}(\theta)), \quad \oint_{\partial C} \vec{F} \cdot d\vec{s}$$

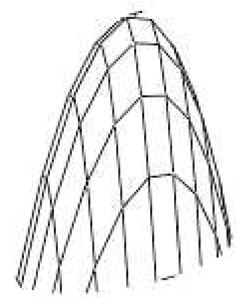
$\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 8)$

$\vec{v} = (-4\sin\theta, 4\cos\theta, 0)$ oriented correctly

$\vec{F}(\vec{r}(\theta)) = (yz, -xz, z) = (32\sin\theta, -32\cos\theta, 8)$

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (-128\sin^2\theta - 128\cos^2\theta)\, d\theta = \int_0^{2\pi} (-128)\, d\theta = -256\pi$$

15. (15 points) A rectangular solid sits on the xy-plane with its top four vertices on the paraboloid $z = 4 - x^2 - 4y^2$. Find the dimensions and volume of the largest such box.



The dimensions are $2x$, $2y$ and z.

$V = (2x)(2y)z = 4xyz = 4xy(4 - x^2 - 4y^2) = 16xy - 4x^3y - 16xy^3$

$V_x = 16y - 12x^2y - 16y^3 = 0$ $\qquad$ $V_y = 16x - 4x^3 - 48xy^2 = 0$

$x > 0$, $y > 0$ and $z > 0$ to give positive, non-zero dimensions.

(1) $4 - 3x^2 - 4y^2 = 0$ $\qquad$ (2) $4 - x^2 - 12y^2 = 0$

3∗(1) - (2): $8 - 8x^2 = 0 \qquad x^2 = 1 \qquad x = 1$

$4y^2 = 4 - 3x^2 = 1 \qquad y^2 = \frac{1}{4} \qquad y = \frac{1}{2}$

$z = 4 - x^2 - 4y^2 = 4 - 1 - 1 = 2$

The dimensions are $2x = 2$, $2y = 1$ and $z = 2$. $\qquad V = (2x)(2y)z = 4$