Name________________________ Sec__________

MATH 251 Honors Exam 1 Spring 2011

Section 500 Solutions P. Yasskin

1-10	/50	14	/11
11	/11	15	/11
12	/11		
13	/11	Total	/105

Multiple Choice: (5 points each. No part credit.)

1. Consider the line $X = P + t\vec{v}$ where $P = (2,3,2)$ and $\vec{v} = (2,-1,2)$.
Drop a perpendicular from the point $Q = (-1,0,5)$ to a point R on the line. Then $R =$
HINT: Draw a figure.

a. $\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$

b. $\left(\frac{8}{3},\frac{8}{3},\frac{8}{3}\right)$ Correct Choice

c. $\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)$

d. $(4,2,4)$

e. $\left(\frac{8}{3},\frac{10}{3},\frac{8}{3}\right)$

$\overrightarrow{PQ} = Q - P = (-3,-3,3)$ $\quad \text{proj}_{\vec{v}}\overrightarrow{PQ} = \dfrac{\overrightarrow{PQ}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \dfrac{-6+3+6}{9}(2,-1,2) = \left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)$

$R = P + \text{proj}_{\vec{v}}\overrightarrow{PQ} = \left(\frac{8}{3},\frac{8}{3},\frac{8}{3}\right)$

2. If $\vec{u}$ is 5 cm long and points $30°$ WEST of NORTH and $\vec{v}$ is 4 cm long and points $30°$ EAST of NORTH, then $\vec{u}\times\vec{v}$ is

a. 10 cm long and points DOWN.

b. 10 cm long and points UP.

c. 10 cm long and points SOUTH.

d. $10\sqrt{3}$ cm long and points DOWN. Correct Choice

e. $10\sqrt{3}$ cm long and points SOUTH.

Since the angle between the vectors is $\theta = 60°$, the length is

$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta = 5\cdot 4\cdot\dfrac{\sqrt{3}}{2} = 10\sqrt{3}$

Put your right hand fingers pointing $30°$ WEST of NORTH with the palm facing $30°$ EAST of NORTH, then your thumb points DOWN.

3. Find the point where the line $(x,y,z) = (3-2t, 2-t, 1+t)$ intersects the plane $x+y+3z = 2$.
At this point, $x+y+z =$

a. 2

b. 4

c. 6

d. 8

e. The line does not intersect the plane. Correct Choice

Substitute the line into the plane: $(3-2t)+(2-t)+3(1+t) = 2$ or $8 = 2$
which is impossible. So the line does not intersect the plane.

4. The graph of the equation $x^2 + 4x - y^2 + 4y + z^2 + 2z = -1$ is a

- **a**. hyperboloid of one sheet
- **b**. hyperboloid of two sheets
- **c**. cone Correct Choice
- **d**. hyperbolic paraboloid
- **e**. hyperbolic cylinder

x^2, y^2, and z^2 are all present with two +'s and one –. So this is a hyperboloid or cone.

Complete the squares to get $(x+2)^2 - (y-2)^2 + (z+1)^2 = 0$ which is a cone.

5. For the helix $\vec{r}(t) = (3t, \sin(4t), \cos(4t))$, which of the following is FALSE?

- **a**. $\vec{v} = (3, 4\cos(4t), -4\sin(4t))$
- **b**. $\vec{a} = (0, -16\sin(4t), -16\cos(4t))$
- **c**. $\vec{j} = (0, -64\cos(4t), 64\sin(4t))$
- **d**. speed $= 25$ Correct Choice
- **e**. arclength between $(0,0,1)$ and $(3\pi, 0, 1)$ is 5π

speed $= |\vec{v}| = \sqrt{9 + 16\cos^2(4t) + 16\sin^2(4t)} = \sqrt{25} = 5$

6. For the helix $\vec{r}(t) = (3t, \sin(4t), \cos(4t))$, which of the following is FALSE?

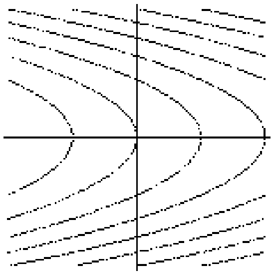
- **a**. $\hat{T} = \left(\frac{3}{5}, \frac{4}{5}\cos(4t), -\frac{4}{5}\sin(4t)\right)$
- **b**. $\hat{N} = (0, -\sin(4t), -\cos(4t))$
- **c**. $\hat{B} = \left(-\frac{4}{5}, -\frac{3}{5}\cos(4t), -\frac{3}{5}\sin(4t)\right)$ Correct Choice
- **d**. $a_T = 0$
- **e**. $a_N = 16$

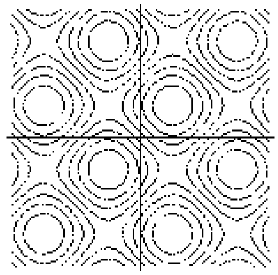
$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4\cos(4t) & -4\sin(4t) \\ 0 & -16\sin(4t) & -16\cos(4t) \end{vmatrix} = 16(-4, 3\cos(4t), -3\sin(4t))$$

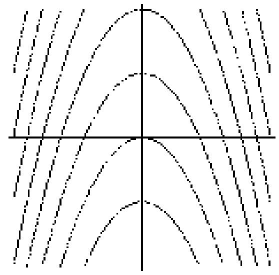
$$|\vec{v} \times \vec{a}| = 16\sqrt{16 + 9\cos^2(4t) + 9\sin^2(4t)} = 80$$

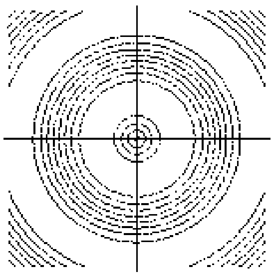
$$\hat{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \left(-\frac{4}{5}, \frac{3}{5}\cos(4t), -\frac{3}{5}\sin(4t)\right)$$

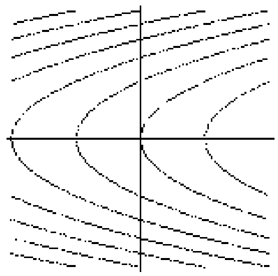
7. Which of the following is the contour plot of $f(x,y) = y^2 + x + 1$?

a. 

b. 

c. 

d. 

e. 

(a) Correct Choice

The contours are $y^2 + x + 1 = C$ or $x = C - 1 - y^2$ which are parabolas opening left.

8. If $P(2,3) = 5$ and $\dfrac{\partial P}{\partial x}(2,3) = 0.4$ and $\dfrac{\partial P}{\partial y}(2,3) = -0.3$, estimate $P(2.1, 2.8)$.

a. 4.9
b. 4.98
c. 4.99
d. 5.01
e. 5.1 Correct Choice

$$P(x,y) = P(2,3) + P_x(2,3)(x-2) + P_y(2,3)(y-3)$$

$$P(2.1, 2.8) = P(2,3) + P_x(2,3)(2.1-2) + P_y(2,3)(2.8-3)$$

$$= 5 + .4(.1) - .3(-.2) = 5.1$$

9. Currently for a certain box, the length L is 5 cm and increasing at 0.2 cm/sec, the width W is 4 cm and decreasing at 0.3 cm/sec, the height H is 3 cm and increasing at 0.1 cm/sec. Then currently, the volume V is

a. increasing at 0.1 cm/sec.
b. decreasing at 0.1 cm/sec. Correct Choice
c. increasing at 0.2 cm/sec.
d. decreasing at 0.2 cm/sec.
e. increasing at 0.3 cm/sec.

$$V = LWH \qquad \frac{dL}{dt} = 0.2 \qquad \frac{dW}{dt} = -0.3 \qquad \frac{dH}{dt} = 0.1$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial L}\frac{dL}{dt} + \frac{\partial V}{\partial W}\frac{dW}{dt} + \frac{\partial V}{\partial H}\frac{dH}{dt} = WH\frac{dL}{dt} + LH\frac{dW}{dt} + LW\frac{dH}{dt}$$

$$= 4 \cdot 3 \cdot 0.2 - 5 \cdot 3 \cdot 0.3 + 5 \cdot 4 \cdot 0.1 = -0.1$$

10. The temperature of a frying pan is $T = \dfrac{1}{1+x^2+4y^2}$. An ant is located at $(2,1)$. In what **unit vector** direction should the ant move to **decrease** the temperature as fast as possible?

a. $(-1,-2)$

b. $(1,2)$ Part Credit

c. $(1,-2)$

d. $\left(\dfrac{-1}{\sqrt{5}}, \dfrac{-2}{\sqrt{5}}\right)$ Part Credit

e. $\left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$ Correct Choice

$$\vec{\nabla}T = \left(\frac{-2x}{(1+x^2+4y^2)^2}, \frac{-8y}{(1+x^2+4y^2)^2}\right) \qquad \vec{\nabla}T\Big|_{(2,1)} = \left(\frac{-4}{81}, \frac{-8}{81}\right)$$

$$\left|\vec{\nabla}T\right| = \frac{\sqrt{16+64}}{81} = \frac{\sqrt{80}}{81} = \frac{4\sqrt{5}}{81}$$

Maximum decrease is in the direction of $\vec{v} = -\vec{\nabla}T\Big|_{(2,1)} = \left(\dfrac{4}{81}, \dfrac{8}{81}\right)$.

The unit vector is $\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{81}{4\sqrt{5}}\left(\dfrac{4}{81}, \dfrac{8}{81}\right) = \left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (11 points) Find the mass of the helical wire $\vec{r}(t) = (3t, \sin(4t), \cos(4t))$ from $(0,0,1)$ to $(3\pi,0,1)$ if its linear density is $\rho = x^2+y^2+z^2$.

$$\vec{v} = (3, 4\cos(4t), -4\sin(4t)) \qquad |\vec{v}| = \sqrt{9+16\cos^2(4t)+16\sin^2(4t)} = 5$$

$$\rho(\vec{r}(t)) = 9t^2 + \sin^2(4t) + \cos^2(4t) = 9t^2 + 1$$

$$M = \int_{(0,0,1)}^{(3\pi,0,1)} \rho\, ds = \int_0^\pi \rho(\vec{r}(t))\,|\vec{v}|\, dt = \int_0^\pi (9t^2+1)\,5\, dt = 5\left[3t^3 + t\right]_0^\pi = 5(3\pi^3 + \pi)$$

12. (11 points) A bead slides along the helix $\vec{r}(t) = (3t, \sin(4t), \cos(4t))$ from $(0,0,1)$ to $(3\pi,0,1)$ under the action of the force $\vec{F} = (x, xy, xz)$. Find the work done.

$$\vec{v} = (3, 4\cos(4t), -4\sin(4t)) \qquad \vec{F}(\vec{r}(t)) = (3t, 3t\sin(4t), 3t\cos(4t))$$

$$\vec{F}\cdot\vec{v} = 9t + 12t\cos(4t)\sin(4t) - 12t\sin(4t)\cos(4t) = 9t$$

$$W = \int_{(0,0,1)}^{(3\pi,0,1)} \vec{F}\cdot d\vec{s} = \int_0^\pi \vec{F}\cdot\vec{v}\, dt = \int_0^\pi 9t\, dt = \left[\frac{9t^2}{2}\right]_0^\pi = \frac{9}{2}\pi^2$$

13. (11 points) Find the volume of the parallelepiped with edges $\vec{u} = (2,0,1,0)$, $\vec{v} = (0,3,2,0)$ and $\vec{w} = (1,0,0,2)$ in $\mathbb{R}^4$.

$$\perp(\vec{u},\vec{v},\vec{w}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{l} \\ 2 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix} - \hat{l}\begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$=(-6,-8,12,3)$$

$$V = |\perp(\vec{u},\vec{v},\vec{w})| = \sqrt{36+64+144+9} = \sqrt{253}$$

14. (11 points) Find the plane tangent to the level surface $x\sin z + y\cos z = 3$ at the point $(x,y,z) = \left(3,2,\frac{\pi}{2}\right)$. Find the z-intercept.

$$F(x,y,z) = x\sin z + y\cos z \qquad \vec{\nabla}F = (\sin z, \cos z, x\cos z - y\sin z) \qquad \vec{N} = \vec{\nabla}F\Big|_{(3,2,\pi/2)} = (1,0,-2)$$

$$\vec{N}\cdot X = \vec{N}\cdot P \qquad x - 2z = 3 - 2\left(\frac{\pi}{2}\right) = 3 - \pi$$

The plane is $x - 2z = 3 - \pi$ or $z = \frac{1}{2}(x + \pi - 3) = \frac{1}{2}x + \frac{\pi}{2} - \frac{3}{2}$. The z-intercept is $\frac{\pi}{2} - \frac{3}{2}$.

15. (11 points) Determine whether or not each of these limits exists. If it exists, find its value.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+3y^6}$

Approach along the x-axis:

$$\lim_{\substack{y=0\\x\to0}} \frac{xy^3}{x^2+3y^6} = \lim_{x\to0}\frac{0}{3y^6} = 0$$

Approach along $x = y^3$:

$$\lim_{\substack{x=y^3\\y\to0}} \frac{xy^3}{x^2+3y^6} = \lim_{x\to0}\frac{y^6}{4y^6} = \frac{1}{4}$$

Limit Does Not Exist

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{(x^2+y^2)^2}$

Use polar coordinates $x = r\cos\theta$ $y = y\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{(x^2+y^2)^2} = \lim_{r\to0}\frac{r^6\cos^6\theta + r^6\sin^6\theta}{r^4} = \lim_{r\to0} r^2(\cos^6\theta + \sin^6\theta) = 0$$

because $\cos^6\theta + \sin^6\theta$ is bounded between 0 and 1.

Limit Exists with value 0.

c. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+xy^2}{x+x^3}$

Factor x out of the top and bottom. Then evaluate:

$$\lim_{(x,y)\to(0,0)} \frac{x+xy^2}{x+x^3} = \lim_{(x,y)\to(0,0)} \frac{x(1+y^2)}{x(1+x^2)} = \lim_{(x,y)\to(0,0)} \frac{1+y^2}{1+x^2} = 1$$

Limit Exists with value 1.