Name______________________ Sec________

1-12	/48	14	/16
13	/16	15	/25
		Total	/105

MATH 251 Final Exam Spring 2011

Section 200/511 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Find the projection of the vector $\vec{u} = (12,-3,4)$ along the vector $\vec{v} = (2,1,-2)$.

a. $\left(\frac{-12}{13}, \frac{3}{13}, \frac{-4}{13}\right)$

b. $\left(\frac{12}{13}, \frac{-3}{13}, \frac{4}{13}\right)$

c. $\left(\frac{-26}{9}, \frac{-13}{9}, \frac{26}{9}\right)$

d. $\left(\frac{26}{9}, \frac{13}{9}, \frac{-26}{9}\right)$ Correct Choice

e. $\left(\frac{108}{13}, \frac{-9}{13}, \frac{36}{13}\right)$

$\vec{u}\cdot\vec{v} = 24-3-8 = 13 \qquad |\vec{v}| = \sqrt{4+1+4} = 3$

$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{13}{9}(2,1,-2) = \left(\frac{26}{9}, \frac{13}{9}, \frac{-26}{9}\right)$

2. If $|\vec{u}| = 4$ and $|\vec{v}| = 3$ and the angle between $\vec{u}$ and $\vec{v}$ is $\theta = \frac{\pi}{3}$ then $|\vec{u}\times\vec{v}| =$

a. $6\sqrt{3}$ Correct Choice

b. 6

c. $3\sqrt{3}$

d. 3

e. $12\sqrt{3}$

$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta = 4\cdot3\cdot\frac{\sqrt{3}}{2} = 6\sqrt{3}$

3. Find the point where the line through the origin perpendicular to the plane $3x-3y+4z = 17$ intersects that plane. At this point $x+y+z =$

a. 2 Correct Choice

b. 3

c. 5

d. 17

e. $\frac{1}{2}$

The normal to the plane is: $\vec{N} = (3,-3,4)$.

The line through the origin O in the direction $\vec{N}$ is $X = O + t\vec{N}$ or $(x,y,z) = (3t,-3t,4t)$.

Plug the line into the plane and solve for t: $2x+3y+4z = 3(3t)-3(-3t)+4(4t) = 34t = 17$.

So $t = \frac{1}{2}$. Plug into the line: $(x,y,z) = \left(\frac{3}{2}, \frac{-3}{2}, 2\right)$ So $x+y+z = 2$

4. Find the arc length of the curve $\vec{r}(t) = (t^2, 2t, \ln t)$ between $(1,2,0)$ and $(e^2, 2e, 1)$.

- **a.** e^2 Correct Choice
- **b.** $e^2 + 1$
- **c.** $e^2 - 1$
- **d.** $\frac{2}{3}e^3 + e$
- **e.** $\frac{2}{3}e^3 + e - \frac{5}{3}$

$\vec{r}(t) = (t^2, 2t, \ln t)$ $\quad \vec{v} = \left(2t, 2, \frac{1}{t}\right)$ $\quad |\vec{v}| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\left(2t + \frac{1}{t}\right)^2} = 2t + \frac{1}{t}$

$L = \int_{(1,2,0)}^{(e^2,e,1)} ds = \int_1^e |\vec{v}|\, dt = \int_1^e \left(2t + \frac{1}{t}\right) dt = \left[t^2 + \ln t\right]_1^e = (e^2 + 1) - (1) = e^2$

5. The pressure, P, density, ρ, and temperature, T, of a certain ideal gas are related by $P = 3\rho T$.
At the point $(1,2,3)$, the density and its gradient are $\rho = 4$ and $\vec{\nabla}\rho = (0.2, 0.4, 0.1)$,
while the temperature and its gradient are $T = 300$ and $\vec{\nabla}T = (2,1,3)$.
Hence the pressure is $P = 3 \cdot 4 \cdot 300 = 3600$ and its gradient is $\vec{\nabla}P =$

- **a.** $(1802.4, 2704.8, 2701.2)$
- **b.** $(600.8, 901.900.4)$
- **c.** $(68, 132, 42)$
- **d.** $(204, 372, 126)$ Correct Choice
- **e.** $(1.2, 1.2, 0.9)$

$$\frac{\partial P}{\partial x} \approx \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial x} = 3T\frac{\partial \rho}{\partial x} + 3\rho\frac{\partial T}{\partial x} = 3 \cdot 300 \cdot 0.2 + 3 \cdot 4 \cdot 2 = 204$$

$$\frac{\partial P}{\partial y} \approx \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial y} = 3T\frac{\partial \rho}{\partial y} + 3\rho\frac{\partial T}{\partial y} = 3 \cdot 300 \cdot 0.4 + 3 \cdot 4 \cdot 1 = 372$$

$$\frac{\partial P}{\partial z} \approx \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial z} = 3T\frac{\partial \rho}{\partial z} + 3\rho\frac{\partial T}{\partial z} = 3 \cdot 300 \cdot 0.1 + 3 \cdot 4 \cdot 3 = 126$$

6. Find the equation of the plane tangent to the graph of $x^2yz^2 - 2y^2z^3 = 10$ at the point $(3,2,1)$.
The z-intercept is

- **a.** $\frac{6}{25}$
- **b.** 5
- **c.** $\frac{25}{6}$ Correct Choice
- **d.** 60
- **e.** $\frac{5}{3}$

$F(x,y,z) = x^2yz^2 - 2y^2z^3$ $\quad \vec{\nabla}F = (2xyz^2, x^2z^2 - 4yz^3, 2x^2yz - 6y^2z^2)$

$\vec{N} = \vec{\nabla}F(3,2,1) = \left(2(3)(2), (3)^2 - 4(2), 2(3)^2(2) - 6(2)^2\right) = (12, 1, 12)$

$\vec{N} \cdot X = \vec{N} \cdot P$ $\quad 12x + y + 12z = 12(3) + (2) + 12(1) = 50$ $\quad z$-intercept $= \frac{50}{12} = \frac{25}{6}$

7. Han Deut is flying the Millennium Eagle through a dangerous zenithon field whose density is $\rho = xyz$. If his current position is $(x, y, z) = (1, -1, 2)$, in what **unit** vector direction should he travel to **decrease** the density as fast as possible?

a. $(2, -2, 1)$

b. $\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ Correct Choice

c. $(-2, 2, -1)$

d. $\left(\frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$

e. $\left(\frac{-2}{3}, \frac{-2}{3}, \frac{-1}{3}\right)$

$\vec{\nabla}\rho = (yz, xz, xy)$ $\quad \vec{\nabla}\rho\Big|_{(1,-1,2)} = (-2, 2, -1)$ $\quad \left|\vec{\nabla}\rho\right| = \sqrt{4+4+1} = 3$

He should travel in the direction $\vec{v} = -\vec{\nabla}\rho = (2, -2, 1)$ or the unit direction $\hat{v} = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$.

8. Compute $\iint e^{-x^2-y^2}\,dA$ over the quarter circle $x^2 + y^2 \le 9$ in the first quadrant.

a. $\frac{\pi}{2}e^{-9}$

b. $\frac{\pi}{2}(e^{-9} - 1)$

c. $-\frac{\pi}{4}e^{-9}$

d. $\frac{\pi}{4}(1 - e^{-9})$ Correct Choice

e. $\pi(1 - e^{-9})$

$$\iint e^{-x^2-y^2}\,dA = \int_0^{\pi/2}\int_0^3 e^{-r^2}\,r\,dr\,d\theta = \frac{\pi}{2}\left[\frac{e^{-r^2}}{-2}\right]_0^3 = \frac{\pi}{4}(1 - e^{-9})$$

9. Compute $\int_0^8 \int_{x^{1/3}}^2 \cos(y^4)\,dy\,dx$

HINT: Reverse the order of integration.

a. $\frac{1}{4}\sin(4) - \frac{1}{4}$

b. $\frac{1}{4}\sin(64) - \frac{1}{4}$

c. $\frac{1}{4}\sin(64)$

d. $\frac{1}{4}\sin(16) - \frac{1}{4}$

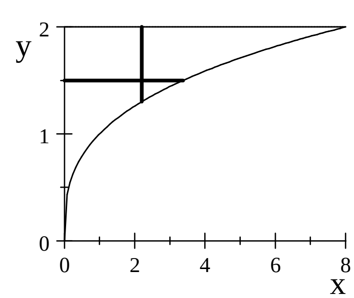
e. $\frac{1}{4}\sin(16)$ Correct Choice

Plot the region. Reverse the order.

Compute new limits: $y = x^{1/3} \quad \Rightarrow \quad x = y^3$

$$\int_0^8 \int_{x^{1/3}}^2 \cos(y^4)\,dy\,dx = \int_0^2 \int_0^{y^3} \cos(y^4)\,dx\,dy = \int_0^2 \cos(y^4)\left[x\right]_{x=0}^{y^3} dy$$

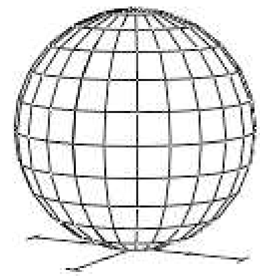
$$= \int_0^2 y^3\cos(y^4)\,dy = \left[\frac{\sin(y^4)}{4}\right]_{y=o}^2 = \frac{1}{4}\sin(16)$$



10. Compute $\iiint z\,dV$ over the solid sphere $x^2 + y^2 + (z-1)^2 \le 1$

given in spherical coordinates by $\rho = 2\cos\varphi$.

HINT: The whole sphere has $z \ge 0$. What does this say about φ?



a. $\dfrac{2\pi}{3}$

b. $\dfrac{4\pi}{3}$ Correct Choice

c. $\dfrac{4\pi}{6}$

d. $\dfrac{\pi}{3}$

e. $\dfrac{\pi}{12}$

$z = \rho\cos\varphi \qquad dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$

$$\iiint z\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^{2\cos\varphi} \rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = 2\pi\int_0^{\pi/2}\left[\frac{\rho^4}{4}\right]_0^{2\cos\varphi}\cos\varphi\sin\varphi\,d\varphi$$

$$= 2\pi\int_0^{\pi/2} 4\cos^5\varphi\sin\varphi\,d\varphi = 8\pi\left[-\frac{\cos^6\varphi}{6}\right]_0^{\pi/2} = \frac{4\pi}{3}$$

11. Compute $\int \vec{F}\cdot d\vec{s}$ for $\vec{F} = (2x, 2y, 2z)$ along the curve $\vec{r}(t) = \left(\frac{2}{t}, \frac{4}{t}, \frac{6}{t}\right)$

from $(2,4,6)$ to $(1,2,3)$.

HINT: Find a scalar potential.

a. -70

b. -42 Correct Choice

c. 0

d. 42

e. 70

$\vec{F} = \vec{\nabla} f \qquad \partial_x f = 2x \qquad \partial_y f = 2y \qquad \partial_z f = 2z \quad \Rightarrow \quad f(x,y,z) = x^2 + y^2 + z^2$

$$\int \vec{F}\cdot d\vec{s} = \int \vec{\nabla} f\cdot d\vec{s} = f(1,2,3) - f(2,4,6) = (1^2+2^2+3^2) - (2^2+4^2+6^2) = -42$$

12. Compute $\oint \vec{F}\cdot d\vec{s}$ for $\vec{F} = (\sec(x^3) - 5y, \cos(y^5) + 3x)$ counterclockwise around the triangle with vertices $(0,0)$, $(8,0)$ and $(0,4)$.

Hint: Use Green's Theorem.

a. 12

b. 16

c. 32

d. 64

e. 128 Correct Choice

$P = \sec(x^3) - 5y \qquad Q = \cos(y^5) + 3x \qquad \partial_x Q - \partial_y P = 3 - (-5) = 8$

$$\oint \vec{F}\cdot d\vec{s} = \oint P\,dx + Q\,dy = \iint \partial_x Q - \partial_y P\,dx\,dy = \iint 8\,dx\,dy = 8\text{Area} = 8\cdot\frac{1}{2}\cdot 8\cdot 4 = 128$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (16 points) Use Lagrange multipliers to find 4 numbers, a, b, c, and d, whose product is $\frac{2}{3}$ and for which $a+2b+3c+4d$ is a minimum.

Minimize $f = a+2b+3c+4d$ subject to the constraint $g = abcd = \frac{2}{3}$.

$\vec{\nabla} f = (1,2,3,4) \qquad \vec{\nabla} g = (bcd, acd, abd, abc)$

$\vec{\nabla} f = \lambda \vec{\nabla} g$: $\qquad 1 = \lambda bcd \qquad 2 = \lambda acd \qquad 3 = \lambda abd \qquad 4 = \lambda abc$

Make the right sides all be $\lambda abcd$ and equate: $\quad \lambda abcd = a = 2b = 3c = 4d$

So $\quad b = \frac{a}{2} \qquad c = \frac{a}{3} \qquad d = \frac{a}{4} \qquad$ Substitute into the constraint:

$$abcd = a\frac{a}{2}\frac{a}{3}\frac{a}{4} = \frac{a^4}{24} = \frac{2}{3} \quad \Rightarrow \quad a^4 = \frac{2}{3} \cdot 24 = 16 \quad \Rightarrow \quad a = 2,\, b = 1,\, c = \frac{2}{3},\, d = \frac{1}{2}$$

14. (16 points) Find the mass and the y-component of the center of mass of the quarter of the cylinder $x^2+y^2 \le 4$ with $0 \le z \le 3$ in the first octant $(x \ge 0, y \ge 0, z \ge 0)$ if the mass density is $\delta = xyz$.

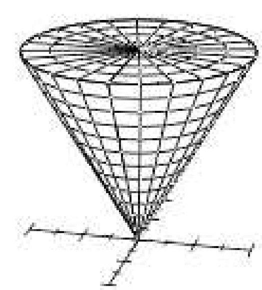
$$\delta = xyz = r\cos\theta\, r\sin\theta\, z = r^2\sin\theta\cos\theta\, z \qquad dV = r\,dr\,d\theta\,dz$$

$$M = \iiint \delta\,dV = \int_0^3\int_0^{\pi/2}\int_0^2 r^3\sin\theta\cos\theta\, z\,dr\,d\theta\,dz = \left[\frac{r^4}{4}\right]_0^2\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}\left[\frac{z^2}{2}\right]_0^3 = 4 \cdot \frac{1}{2} \cdot \frac{9}{2} = 9$$

$$M_{xz} = \iiint y\delta\,dV = \int_0^3\int_0^{\pi/2}\int_0^2 r^4\sin^2\theta\cos\theta\, z\,dr\,d\theta\,dz = \left[\frac{r^5}{5}\right]_0^2\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2}\left[\frac{z^2}{2}\right]_0^3 = \frac{32}{5} \cdot \frac{1}{3} \cdot \frac{9}{2} = \frac{48}{5}$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{48}{5}\frac{1}{9} = \frac{16}{15}$$

15. (25 points) Verify Gauss' Theorem $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\,dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$



for the vector field $\vec{F} = (2xy^2, 2yx^2, z(x^2+y^2))$ and the solid above the cone $z = 2\sqrt{x^2+y^2}$ below the plane $z = 2$.

Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence:

$$\vec{\nabla} \cdot \vec{F} = 2y^2 + 2x^2 + x^2 + y^2 = 3x^2 + 3y^2$$

b. Express the divergence and the volume element in the appropriate coordinate system:

$$\vec{\nabla} \cdot \vec{F} = 3r^2 \qquad\qquad dV = r\,dr\,d\theta\,dz$$

c. Find the limits of integration:

$0 \le \theta \le 2\pi \qquad 2\sqrt{x^2+y^2} \le z \le 2$ becomes $2r \le z \le 2 \qquad 2r = 2$ when $r = 1$

d. Compute the left hand side:

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\,dV = \int_0^{2\pi}\int_0^1\int_{2r}^2 (3r^2) r\,dz\,dr\,d\theta = 2\pi\int_0^1 3r^3\big[z\big]_{2r}^2\,dr = 6\pi\int_0^1 r^3(2-2r)\,dr$$

$$= 12\pi\int_0^1 (r^3 - r^4)\,dr = 12\pi\left[\frac{r^4}{4} - \frac{r^5}{5}\right]_0^1 = 12\pi\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{3\pi}{5}$$

Second the Right Hand Side:

The boundary surface consists of a cone C and a disk D with appropriate orientations.

e. Complete the parametrization of the cone C:

$$\vec{R}(r,\theta) = \left(r\cos\theta, r\sin\theta, \underline{\quad 2r \quad}\right)$$

f. Compute the tangent vectors:

$$\vec{e}_r = (\cos\theta, \sin\theta, 2)$$
$$\vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

g. Compute the normal vector:

$$\vec{N} = \hat{\imath}(-2r\cos\theta) - \hat{\jmath}(--2r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (-2r\cos\theta, -2r\sin\theta, r)$$

This is up and in. Reverse $\vec{N} = (2r\cos\theta, 2r\sin\theta, -r)$

h. Evaluate $\vec{F} = (2xy^2, 2yx^2, z(x^2+y^2))$ on the cone:

$$\vec{F}\Big|_{\vec{R}(r,\theta)} = (2r^3\cos\theta\sin^2\theta, 2r^3\sin\theta\cos^2\theta, 2r^3)$$

i. Compute the dot product:

$$\vec{F}\cdot\vec{N} = 4r^4\cos^2\theta\sin^2\theta + 4r^4\sin^2\theta\cos^2\theta - 2r^4 = 2r^4(4\sin^2\theta\cos^2\theta - 1)$$

j. Compute the flux through C:

$$\iint_C \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 2r^4(4\sin^2\theta\cos^2\theta - 1)\,dr\,d\theta = 2\left[\frac{r^5}{5}\right]_0^1 \int_0^{2\pi}(\sin^2(2\theta) - 1)\,d\theta$$

$$= \frac{2}{5}\int_0^{2\pi}(-\cos^2(2\theta))\,d\theta = -\frac{2}{5}\int_0^{2\pi}\frac{1-\cos(4\theta)}{2}\,d\theta = -\frac{1}{5}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{2\pi} = -\frac{2\pi}{5}$$

k. Complete the parametrization of the disk D:

$$\vec{R}(r,\theta) = \left(r\cos\theta, r\sin\theta, \underline{\quad 2 \quad}\right)$$

l. Compute the tangent vectors:

$$\vec{e}_r = (\cos\theta, \sin\theta, 0)$$
$$\vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

m. Compute the normal vector:

$$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (0,0,r)$$

This is correctly up.

n. Evaluate $\vec{F} = (2xy^2, 2yx^2, z(x^2+y^2))$ on the disk:

$$\vec{F}\Big|_{\vec{R}(r,\theta)} = (2r^3\cos\theta\sin^2\theta, 2r^3\sin\theta\cos^2\theta, 2r^2)$$

o. Compute the dot product:

$$\vec{F}\cdot\vec{N} = 2r^3$$

p. Compute the flux through D:

$$\iint_D \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 2r^3\,dr\,d\theta = 2\pi\left[\frac{r^4}{2}\right]_0^1 = \pi$$

q. Compute the right hand side:

$$\iint_{\partial V}\vec{F}\cdot d\vec{S} = \iint_C \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S} = -\frac{2\pi}{5} + \pi = \frac{3\pi}{5}$$ which agrees with (d).