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MATH 251 Final Exam Fall 2012

Sections 515 Solutions P. Yasskin

1-9	/45
10	/15
11	/25
12	/15
Total	/100

Multiple Choice: (5 points each. No part credit.)

1. Points A , B , C and D are the vertices of a parallelogram traversed in order.

If $A = (2, 4, 1)$, $B = (3, -2, 0)$ and $D = (1, 3, -2)$, then $C =$

- a. $(2, -3, -3)$ Correct Choice
- b. $(4, -1, 3)$
- c. $(0, 9, -1)$
- d. $(6, 5, -1)$
- e. $(4, \frac{9}{2}, 0)$

SOLUTION: $\vec{AD} = D - A = (1, 3, -2) - (2, 4, 1) = (-1, -1, -3)$

$C = B + \vec{AD} = (3, -2, 0) + (-1, -1, -3) = (2, -3, -3)$

2. Which vector is perpendicular to the surface $x^2z^3 + y^3z^2 = 1$ at the point $(3, -2, 1)$?

- a. $(12, -24, 43)$
- b. $(6, -12, 43)$
- c. $(6, 12, 43)$
- d. $(6, -12, 11)$
- e. $(-12, -24, -22)$ Correct Choice

SOLUTION: $f = x^2z^3 + y^3z^2$ $\vec{\nabla}f = (2xz^3, 3y^2z^2, 3x^2z^2 + 2y^3z)$ $\vec{\nabla}f|_{(3,-2,1)} = (6, 12, 11)$

$(-12, -24, -22) = -2(6, 12, 11)$

3. Find the point on the elliptic paraboloid $\vec{R}(t, \theta) = (3t \cos \theta, 2t \sin \theta, 1 + t^2)$ where a unit normal is

$$\hat{N} = \left(\frac{-2\sqrt{3}}{5}, \frac{-2}{5}, \frac{3}{5} \right).$$

- a. $\left(\frac{3}{2}, \sqrt{3}, 2 \right)$
- b. $\left(-\frac{3}{2}\sqrt{3}, -1, 2 \right)$
- c. $\left(3, 2\sqrt{3}, 5 \right)$
- d. $\left(3\sqrt{3}, 2, 5 \right)$ Correct Choice
- e. $\left(-3, -2\sqrt{3}, 5 \right)$

SOLUTION: $\vec{e}_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 \cos \theta & 2 \sin \theta & 2t \\ -3t \sin \theta & 2t \cos \theta & 0 \end{vmatrix} \quad \vec{N} = (-4t^2 \cos \theta, -4t^2 \sin \theta, 6t)$

$$|\vec{N}| = \sqrt{16t^4 + 36t^2} = 2t\sqrt{4t^2 + 9} \quad \hat{N} = \left(\frac{-2t \cos \theta}{\sqrt{4t^2 + 9}}, \frac{-2t \sin \theta}{\sqrt{4t^2 + 9}}, \frac{3}{\sqrt{4t^2 + 9}} \right) = \left(\frac{-2\sqrt{3}}{5}, \frac{-2}{5}, \frac{3}{5} \right)$$

$$\sqrt{4t^2 + 9} = 5 \quad t = 2 \quad -4 \cos \theta = -2\sqrt{3} \quad \cos \theta = \frac{\sqrt{3}}{2} \quad \theta = \frac{\pi}{6}$$

$$\vec{R}\left(2, \frac{\pi}{6}\right) = \left(3 \cdot 2 \cos \frac{\pi}{6}, 2 \cdot 2 \sin \frac{\pi}{6}, 1 + 2^2 \right) = \left(6 \frac{\sqrt{3}}{2}, 4 \frac{1}{2}, 5 \right) = \left(3\sqrt{3}, 2, 5 \right)$$

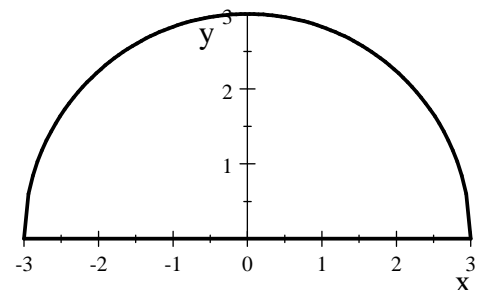
4. Compute $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{1}{1+x^2+y^2} dx dy$

HINT: Plot the region of integration and convert to polar coordinates.

- a. $\frac{\pi}{2} \ln 10$
- b. $\pi \ln 10$ Correct Choice
- c. $\frac{\pi}{2} \arctan 3$
- d. $\pi \arctan 3$
- e. $\pi \arctan 10$

SOLUTION:

$$\begin{aligned} 0 &\leq y \leq 3 \\ -\sqrt{9-y^2} &\leq x \leq \sqrt{9-y^2} \\ \text{or} \\ 0 &\leq r \leq 3 \\ 0 &\leq \theta \leq \pi \end{aligned}$$



$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{1}{1+x^2+y^2} dx dy = \int_0^\pi \int_0^3 \frac{r}{1+r^2} dr d\theta = \left[\frac{\pi}{2} \ln(1+r^2) \right]_0^3 = \frac{\pi}{2} \ln 10$$

5. Find the mass of a wire in the shape of the curve $\vec{r}(t) = (e^t, \sqrt{2}t, e^{-t})$ for $-1 \leq t \leq 1$ if the density is $\rho = x$.

- a. $\frac{e^2}{2} + \frac{e^{-2}}{2}$
 b. $\frac{e^2}{2} - \frac{e^{-2}}{2}$
 c. $\frac{e^2}{2} - \frac{e^{-2}}{2} + 2$ **Correct Choice**
 d. $e^2 - e^{-2}$
 e. $e^2 - e^{-2} + 2$

SOLUTION: $\vec{v} = (e^t, \sqrt{2}, -e^{-t})$ $|\vec{v}| = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}$

$$M = \int \rho ds = \int_{-1}^1 x |\vec{v}| dt = \int_{-1}^1 e^t (e^t + e^{-t}) dt = \int_{-1}^1 (e^{2t} + 1) dt = \left[\frac{e^{2t}}{2} + t \right]_{-1}^1$$

$$= \left(\frac{e^2}{2} + 1 \right) - \left(\frac{e^{-2}}{2} - 1 \right) = \frac{e^2}{2} - \frac{e^{-2}}{2} + 2$$

6. Find the plane tangent to graph of $z = x \cos y + \sin y$ at $(2, \pi)$.
 What is the z -intercept?

- a. $-4 + \pi$
 b. $4 + \pi$
 c. $-4 - \pi$
 d. $4 - \pi$
 e. π **Correct Choice**

SOLUTION: $f(x, y) = x \cos y + \sin y$ $f(2, \pi) = 2 \cos \pi + \sin \pi = -2$

$f_x(x, y) = \cos y$ $f_x(2, \pi) = \cos \pi = -1$

$f_y(x, y) = -x \sin y + \cos y$ $f_y(2, \pi) = -2 \sin \pi + \cos \pi = -1$

$z = f(2, \pi) + f_x(2, \pi)(x - 2) + f_y(2, \pi)(y - \pi)$

tan plane: $z = -2 - 1(x - 2) - 1(y - \pi)$

z -intercept: $x = y = 0$ $z = -2 - 1(-2) - 1(-\pi) = \pi$

7. Find the plane perpendicular to the curve $\vec{r}(t) = (t, t^2, t^3)$ at the point $(1, 1, 1)$.
What is the z -intercept?

- a. 5
b. 4
c. 3
d. 2 Correct Choice
e. 1

SOLUTION: $P = \vec{r}(t) = (1, 1, 1)$ at $t = 1$

$\vec{v}(t) = (1, 2t, 3t^2)$ $\vec{N} = \vec{v}(1) = (1, 2, 3)$

Plane: $\vec{N} \cdot X = \vec{N} \cdot P$ $x + 2y + 3z = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 = 6$

z -intercept: $x = y = 0$ $3z = 6$ $z = 2$

8. Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (y^2, 4xy)$ along the piece of the parabola $y = x^2$ from $(-2, 4)$ to $(2, 4)$ followed by the line segment from $(2, 4)$ back to $(-2, 4)$.
HINT: Use Green's Theorem.

- a. $\frac{256}{5}$ Correct Choice
b. $\frac{768}{5}$
c. $\frac{64}{3}$
d. $\frac{128}{3}$
e. 0

SOLUTION: $\oint \vec{F} \cdot d\vec{s} = \oint P dx + Q dy$ with $P = y^2$ and $Q = 4xy$. By Green's Theorem,

$$\begin{aligned} \oint \vec{F} \cdot d\vec{s} &= \int_{-2}^2 \int_{x^2}^4 (\partial_x Q - \partial_y P) dy dx = \int_{-2}^2 \int_{x^2}^4 (4y - 2y) dy dx = \int_{-2}^2 [y^2]_{x^2}^4 dx \\ &= \int_{-2}^2 (16 - x^4) dx = \left[16x - \frac{x^5}{5} \right]_{-2}^2 = 2 \left(32 - \frac{32}{5} \right) = \frac{256}{5} \end{aligned}$$

9. Compute $\int \vec{F} \cdot d\vec{s}$ for $\vec{F} = (4xy^2, 4x^2y)$ along the line segment from $(1, 2)$ to $(3, 1)$.
HINT: Find a scalar potential.

- a. 4
b. 10 Correct Choice
c. 20
d. 24
e. 26

SOLUTION: $\vec{F} = (4xy^2, 4x^2y) = \vec{\nabla} f$ for $f = 2x^2y^2$ since $\partial_x f = 4xy^2$ and $\partial_y f = 4x^2y$

By the F.T.C.C. $\int \vec{F} \cdot d\vec{s} = \int_{(1,2)}^{(3,1)} \vec{\nabla} f \cdot d\vec{s} = f(3, 1) - f(1, 2) = 2(3)^2(1)^2 - 2(1)^2(2)^2 = 10$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (15 points) A 166 cm piece of wire is cut into 3 pieces of lengths a , b and c .

The piece of length a is folded into a square of side $s = \frac{a}{4}$.

The piece of length b is folded into a rectangle of length $L_1 = \frac{b}{3}$ and width $W_1 = \frac{b}{6}$.

The piece of length c is folded into a rectangle of length $L_2 = \frac{3c}{8}$ and width $W_2 = \frac{c}{8}$.

Find a , b and c so that the total area is a minimum.

What is the total area?

$$\text{Minimize the total area: } A = s^2 + L_1W_1 + L_2W_2 = \frac{a^2}{16} + \frac{b^2}{18} + \frac{3c^2}{64}$$

$$\text{subject to the constraint: } g = a + b + c = 166$$

METHOD 1: Lagrange Multipliers:

$$\vec{\nabla}A = \left(\frac{a}{8}, \frac{b}{9}, \frac{3c}{32} \right) \quad \vec{\nabla}g = (1, 1, 1)$$

$$\vec{\nabla}A = \lambda \vec{\nabla}g \quad \Rightarrow \quad \frac{a}{8} = \lambda \quad \frac{b}{9} = \lambda \quad \frac{3c}{32} = \lambda \quad \Rightarrow \quad \frac{a}{8} = \frac{b}{9} = \frac{3c}{32}$$

$$a = \frac{3c}{4} \quad b = \frac{27c}{32}$$

$$\text{Constraint: } 166 = a + b + c = \frac{3c}{4} + \frac{27c}{32} + c = \frac{83}{32}c \quad \Rightarrow \quad c = 64$$

$$a = \frac{3c}{4} = 48 \quad b = \frac{27c}{32} = 54$$

$$A = \frac{48^2}{16} + \frac{54^2}{18} + \frac{3 \cdot 64^2}{64} = 498$$

METHOD 2: Eliminate a Variable:

$$c = 166 - a - b$$

$$A = \frac{a^2}{16} + \frac{b^2}{18} + \frac{3(166 - a - b)^2}{64}$$

$$A_a = \frac{a}{8} - \frac{3(166 - a - b)}{32} = \frac{7}{32}a + \frac{3}{32}b - \frac{249}{16} = 0 \quad \Rightarrow \quad 7a + 3b = 498 \quad (1)$$

$$A_b = \frac{b}{9} - \frac{3(166 - a - b)}{32} = \frac{3}{32}a + \frac{59}{288}b - \frac{249}{16} = 0 \quad \Rightarrow \quad 3a + \frac{59}{9}b = 498 \quad (2)$$

$$3(1) - 7(2): 9b - 7 \cdot \frac{59}{9}b = -4 \cdot 498 \quad \Rightarrow \quad -\frac{332}{9}b = -1992 \quad \Rightarrow \quad b = 54$$

$$\frac{59}{9}(1) - 3(2): \frac{59}{9} \cdot 7a - 9a = \left(\frac{59}{9} - 3 \right) 498 \quad \Rightarrow \quad \frac{332}{9}a = \frac{5312}{3} \quad \Rightarrow \quad a = 48$$

$$c = 166 - a - b = 166 - 48 - 54 = 64$$

$$A = \frac{48^2}{16} + \frac{54^2}{18} + \frac{3 \cdot 64^2}{64} = 498$$

11. (25 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F} dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$

for the vector field $\vec{F} = (xz, yz, x^2 + y^2)$ and the solid hemisphere $0 \leq z \leq \sqrt{4 - x^2 - y^2}$.



Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence:

$$\vec{\nabla} \cdot \vec{F} = z + z + 0 = 2z$$

b. Express the divergence and the volume element in the appropriate coordinate system:

$$\vec{\nabla} \cdot \vec{F} = 2\rho \cos \varphi \quad dV = \rho^2 \sin \varphi d\rho d\varphi d\theta$$

c. Compute the left hand side:

$$\begin{aligned} \iiint_V \vec{\nabla} \cdot \vec{F} dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 2\rho \cos \varphi \rho^2 \sin \varphi d\rho d\varphi d\theta = 2\pi \int_0^{\pi/2} \cos \varphi \sin \varphi d\varphi \int_0^2 2\rho^3 d\rho \\ &= 2\pi \left[\frac{\sin^2 \varphi}{2} \right]_0^{\pi/2} \left[\frac{\rho^4}{2} \right]_0^2 = 2\pi \left(\frac{1}{2} \right) (8) = 8\pi \end{aligned}$$

Second the Right Hand Side:

The boundary surface consists of a hemisphere H and a disk D with appropriate orientations.

d. Parametrize the disk D :

$$\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, 0)$$

e. Compute the tangent vectors:

$$\vec{e}_r = (\cos \theta, \sin \theta, 0)$$

$$\vec{e}_\theta = (-r \sin \theta, r \cos \theta, 0)$$

f. Compute the normal vector:

$$\vec{N} = \hat{i}(0) - \hat{j}(0) + \hat{k}(r \cos^2 \theta - -r \sin^2 \theta) = (0, 0, r)$$

This is up. Need down. Reverse: $\vec{N} = (0, 0, -r)$

g. Evaluate $\vec{F} = (xz, yz, x^2 + y^2)$ on the disk:

$$\vec{F} \Big|_{\vec{R}(r, \theta)} = (0, 0, r^2)$$

h. Compute the dot product:

$$\vec{F} \cdot \vec{N} = -r^3$$

i. Compute the flux through D :

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 -r^3 dr d\theta = 2\pi \left[-\frac{r^4}{4} \right]_0^2 = -8\pi$$

j. Parametrize the hemisphere H :

$$\vec{R}(\varphi, \theta) = (2 \sin \varphi \cos \theta, 2 \sin \varphi \sin \theta, 2 \cos \varphi)$$

k. Compute the tangent vectors:

$$\vec{e}_\varphi = (2 \cos \varphi \cos \theta, 2 \cos \varphi \sin \theta, -2 \sin \varphi)$$

$$\vec{e}_\theta = (-2 \sin \varphi \sin \theta, 2 \sin \varphi \cos \theta, 0)$$

l. Compute the normal vector:

$$\begin{aligned}\vec{N} &= \hat{i}(4 \sin^2 \varphi \cos \theta) - \hat{j}(-4 \sin^2 \varphi \sin \theta) + \hat{k}(4 \sin \varphi \cos \varphi \sin^2 \theta + 4 \sin \varphi \cos \varphi \cos^2 \theta) \\ &= (4 \sin^2 \varphi \cos \theta, 4 \sin^2 \varphi \sin \theta, 4 \sin \varphi \cos \varphi)\end{aligned}$$

This is oriented correctly as up and out.

m. Evaluate $\vec{F} = (xz, yz, x^2 + y^2)$ on the hemisphere:

$$\vec{F}|_{\vec{R}(\theta, \varphi)} = (4 \sin \varphi \cos \varphi \cos \theta, 4 \sin \varphi \cos \varphi \sin \theta, 4 \sin^2 \varphi)$$

n. Compute the dot product:

$$\begin{aligned}\vec{F} \cdot \vec{N} &= 16 \sin^3 \varphi \cos \varphi \cos^2 \theta + 16 \sin^3 \varphi \cos \varphi \sin^2 \theta + 16 \sin^3 \varphi \cos \varphi \\ &= 16 \sin^3 \varphi \cos \varphi + 16 \sin^3 \varphi \cos \varphi = 32 \sin^3 \varphi \cos \varphi\end{aligned}$$

o. Compute the flux through H :

$$\iint_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi/2} 32 \sin^3 \varphi \cos \varphi d\varphi d\theta = 2\pi [8 \sin^4 \varphi]_0^{\pi/2} = 16\pi$$

p. Compute the **TOTAL** right hand side:

$$\iiint_{\partial V} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} + \iint_H \vec{F} \cdot d\vec{S} = -8\pi + 16\pi = 8\pi \quad \text{which agrees with (c).}$$

12. (15 points) Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for $\vec{F} = (-y, x, z)$

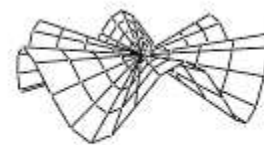
over the "clam shell" surface parametrized by

$$\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r \sin(6\theta))$$

for $r \leq 2$ oriented upward.

HINTS: Use Stokes Theorem.

What is the value of r on the boundary?



SOLUTION:

Boundary: $r = 2$ $\vec{r}(\theta) = (2 \cos \theta, 2 \sin \theta, 2 \sin(6\theta))$ $\vec{v} = (-2 \sin \theta, 2 \cos \theta, 12 \cos(6\theta))$

$$\vec{F}(\vec{r}(\theta)) = (-2 \sin \theta, 2 \cos \theta, 2 \sin(6\theta))$$

$$\vec{F} \cdot \vec{v} = 4 \sin^2 \theta + 4 \cos^2 \theta + 24 \sin(6\theta) \cos(6\theta) = 4 + 24 \sin(6\theta) \cos(6\theta)$$

$$\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \vec{F} \cdot \vec{v} d\theta = \int_0^{2\pi} 4 + 24 \sin(6\theta) \cos(6\theta) d\theta = \left[4\theta + 2 \sin^2(6\theta) \right]_0^{2\pi} = 8\pi$$