1-12	/48
13	/20
14	/10
15	/25
Total	/103

Name______________________

MATH 251 Final Exam Fall 2014

Sections 508 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Find the line of intersection of the planes $3x + 2y - 4z = 8$ and $3x - 2y + 2z = 4$.
This line intersects the xy-plane at

a. $\left(1, -\frac{7}{3}, -3\right)$
b. $(4, 10, 6)$
c. $(1, 2, 0)$
d. $(2, 1, 0)$ Correct Choice
e. $\left(\frac{22}{9}, 2, \frac{4}{3}\right)$

Solution: The desired point satisfies the plane equations and $z = 0$.
So we solve $3x + 2y = 8$ and $3x - 2y = 4$.
Adding the 2 equations: $6x = 12$ $x = 2$ Subtracting: $4y = 4$ $y = 1$

2. The radius of a cylinder is currently $r = 50$ cm and is increasing at $\frac{dr}{dt} = 2\,\frac{\text{cm}}{\text{min}}$.

Its height is currently $h = 100$ cm and decreasing at $\frac{dh}{dt} = -4\,\frac{\text{cm}}{\text{min}}$.

At what rate is the volume changing?

a. $\frac{dV}{dt} = -10\,000\pi$
b. $\frac{dV}{dt} = 10\,000\pi$ Correct Choice
c. $\frac{dV}{dt} = 20\,000\pi$
d. $\frac{dV}{dt} = 30\,000\pi$
e. $\frac{dV}{dt} = 40\,000\pi$

Solution: $V = \pi r^2 h$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt} = 2\pi(50)(100)(2) + \pi(50)^2(-4) = 10\,000\pi$$

3. Find the tangent plane to the graph of $z = \frac{1}{xy}$ at $(1,2)$. The z-intercept is

- **a**. 0
- **b**. $\frac{1}{2}$
- **c**. $\frac{3}{2}$ Correct Choice
- **d**. 2
- **e**. $\frac{5}{2}$

Solution: $f(x,y) = \frac{1}{xy}$ $f_x(x,y) = \frac{-1}{x^2y}$ $f_y(x,y) = \frac{-1}{xy^2}$

$f(1,2) = \frac{1}{2}$ $f_x(1,2) = \frac{-1}{2}$ $f_y(1,2) = \frac{-1}{4}$

$z = f_{\tan}(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$z = \frac{1}{2} - \frac{1}{2}(x-1) - \frac{1}{4}(y-2)$ $c = \frac{1}{2} - \frac{1}{2}(-1) - \frac{1}{4}(-2) = \frac{3}{2}$

4. Find the tangent plane to the graph of the ellipsoid $x^2 + xy + 4y^2 + z^2 = 20$ at $(3,1,2)$. The z-intercept is

- **a**. 10 Correct Choice
- **b**. 20
- **c**. 40
- **d**. 80
- **e**. 160

Solution: $F = x^2 + xy + 4y^2 + z^2$ $P = (3,1,2)$ $\vec{\nabla}F = (2x+y, x+8y, 2z)$

$\vec{N} = \vec{\nabla}F\Big|_P = (7,11,4)$ $\vec{N} \cdot X = \vec{N} \cdot P$ $7x + 11y + 4z = 7(3) + 11(1) + 4(2) = 40$

$4c = 40$ $c = 10$

5. How many critical points does the function $f(x,y) = x^3 - 3xy + y^3$ have?

- **a**. 0
- **b**. 1
- **c**. 2 Correct Choice
- **d**. 3
- **e**. infinitely many

Solution: $f_x = 3x^2 - 3y = 0$ $f_y = -3x + 3y^2 = 0$ $y = x^2$ $x = y^2$

$x = x^4$ $x^4 - x = 0$ $x(x^3 - 1) = 0$ $x = 0$ or 1

If $x = 0$ then $y = 0$. If $x = 1$ then $y = 1$. So 2 critical points.

6. Duke Skywater is flying the Centenial Eagle through a dangerous polaron field whose density is given by $\rho = x^4 + y^3 + z^2$. If Duke is currently at the point $P = (1,2,3)$ and has velocity $\vec{v} = (4,3,2)$, what is the rate of change of the polaron density as seen by Duke at the current instant?

a. 16
b. 24
c. 32
d. 48
e. 64 Correct Choice

Solution: $\vec{\nabla}\rho = (4x^3, 3y^2, 2z)$ $\quad \vec{\nabla}\rho\Big|_{(1,2,3)} = (4,12,6)$

$$\frac{d\rho}{dt} = \vec{v}\cdot\vec{\nabla}\rho = (4,3,2)\cdot(4,12,6) = 16 + 36 + 12 = 64$$

7. Duke Skywater is flying the Centenial Eagle through a dangerous polaron field whose density is given by $\rho = x^4 + y^3 + z^2$. If Duke is currently at the point $P = (1,2,3)$ in what **unit vector** direction should he fly to **REDUCE** the polaron density as fast as possible?

a. $\left(\frac{2}{7}, \frac{6}{7}, \frac{3}{7}\right)$
b. $\left(-\frac{2}{7}, -\frac{6}{7}, -\frac{3}{7}\right)$ Correct Choice
c. $\left(\frac{2}{7}, -\frac{6}{7}, \frac{3}{7}\right)$
d. $\left(\frac{2}{98}, \frac{6}{98}, \frac{3}{98}\right)$
e. $\left(-\frac{2}{98}, -\frac{6}{98}, -\frac{3}{98}\right)$

Solution: $\vec{\nabla}\rho = (4x^3, 3y^2, 2z)$ $\quad \vec{\nabla}\rho\Big|_{(1,2,3)} = (4,12,6)$ $\quad \left|\vec{\nabla}\rho\right| = \sqrt{4^2 + 12^2 + 6^2} = 14$

Fastest DECREASE along: $\vec{u} = -\frac{1}{\left|\vec{\nabla}\rho\right|}\vec{\nabla}\rho\Big|_{(1,2,3)} = \left(-\frac{2}{7}, -\frac{6}{7}, -\frac{3}{7}\right)$

8. Find the maximum value of the function $f = x + 2y + 2z$ on the sphere of radius 5 centered at the origin.

a. 5
b. $\frac{25}{3}$
c. 9
d. $\frac{27}{5}$
e. 15 Correct Choice

Solution: The constraint is $g = x^2 + y^2 + z^2 = 25$.

$\vec{\nabla}f = (1,2,2)$ $\quad \vec{\nabla}g = (2x, 2y, 2z)$ $\quad$ Lagrange equations: $\vec{\nabla}f = \lambda\vec{\nabla}g$

$1 = \lambda 2x \quad 2 = \lambda 2y \quad 2 = \lambda 2z \quad \frac{1}{\lambda} = 2x = y = z$

$g = x^2 + 4x^2 + 4x^2 = 25 \quad 9x^2 = 25 \quad x = \pm\frac{5}{3} \quad y = \pm\frac{10}{3} \quad z = \pm\frac{10}{3}$

For a maximum, we need them positive. $\quad f = \left(\frac{5}{3}\right) + 2\left(\frac{10}{3}\right) + 2\left(\frac{10}{3}\right) = 15$

9. A wire has the shape of the parabola $y = x^2$ which may be parametrized as $\vec{r}(t) = (t, t^2)$ from $(0,0)$ to $\left(\sqrt{6}, 6\right)$. Find it mass if its linear density is $\rho = x$.

a. $\frac{1}{4}\ln\left(2\sqrt{6}+5\right) + \frac{5}{2}\sqrt{6}$
b. $\frac{1}{4}\ln\left(2\sqrt{6}+5\right) - \frac{5}{2}\sqrt{6}$
c. $\frac{125}{12}$
d. $\frac{31}{3}$ Correct Choice
e. $\frac{21}{2}$

Solution: $\vec{v} = (1, 2t)$ $\quad |\vec{v}| = \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2}$ $\quad \rho = x = t$

$M = \int \rho\, ds = \int_0^{\sqrt{6}} \rho |\vec{v}|\, dt = \int_0^{\sqrt{6}} t\sqrt{1+4t^2}\, dt \qquad u = 1+4t^2 \qquad du = 8t\,dt \qquad \frac{1}{8}du = t\,dt$

$M = \frac{1}{8}\int_1^{25} \sqrt{u}\, du = \frac{1}{8}\left[\frac{2u^{3/2}}{3}\right]_1^{25} = \frac{1}{12}(125-1) = \frac{31}{3}$

10. Compute $\int \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2xy^2, 2x^2y)$ along the cubic $y = x^3$ from $(1,1)$ to $(2,8)$.
Hint: Use a theorem.

a. 255 Correct Choice
b. 256
c. 510
d. 60
e. 66

Solution: To use the Fundamental Theorem of Calculus for Curves, we need a scalar potential.

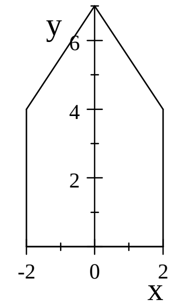
$\vec{F} = \vec{\nabla} f \qquad \partial_x f = 2xy^2 \qquad \partial_y f = 2x^2y \qquad f = x^2y^2$

$\int \vec{F} \cdot d\vec{s} = \int_{(1,1)}^{(2,8)} \vec{\nabla} f \cdot d\vec{s} = f(2,8) - f(1,1) = 2^2 8^2 - 1^2 1^2 = 255$

11. Compute $\oint (3y-2x^2)\,dx + (3y^2-2x)\,dy$

counterclockwise over the complete boundary of the shape at the right, which is a square of side 4 under an isosceles triangle with height 3.

Hint: Use a theorem



a. -110 Correct Choice

b. -22

c. 0

d. 22

e. 110

Solution: Green's Theorem says $\displaystyle\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Here $P = 3y - 2x^2$, $Q = 3y^2 - 2x$ and R is the region inside the square and triangle.

$$\oint_{\partial R} (3y-2x^2)\,dx + (3y^2-2x)\,dy = \iint_R \left(\frac{\partial}{\partial x}(3y^2-2x) - \frac{\partial}{\partial y}(3y-2x^2)\right) dx\,dy$$

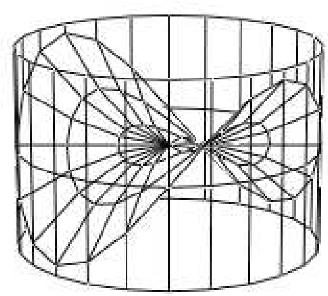
$$= \iint_R ((-2)-(3))\,dx\,dy = -5\iint_R 1\,dx\,dy = -5(Area) = -5\left(4^2 + \frac{1}{2}4 \cdot 3\right) = -110$$

12. Compute $\displaystyle\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the piece of the hyperbolic paraboloid $z = xy$ oriented upward inside the cylinder $x^2 + y^2 = 9$ for the vector field $\vec{F} = \left(-3y, 3x, \sqrt{x^2+y^2}\right)$.

Note, the paraboloid may be parametrized by

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2\sin\theta\cos\theta)$.

Hint: Use a theorem.



a. $\dfrac{27}{2}\pi$

b. 27π

c. 54π Correct Choice

d. 108π

e. 216π

Solution: Stokes' Theorem says $\displaystyle\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial P} \vec{F} \cdot d\vec{s}$.

The boundary of the paraboloid is its intersection with the cylinder where $r = 3$.
So the boundary is the curve $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 9\sin\theta\cos\theta)$.
$\vec{v} = (-3\sin\theta, 3\cos\theta, 9\cos^2\theta - 9\sin^2\theta)$
On the curve $\vec{F} = (-9\sin\theta, 9\cos\theta, 3)$. So $\vec{F} \cdot \vec{v} = 27\sin^2\theta + 27\cos^2\theta + 27\cos^2\theta - 27\sin^2\theta = 54\cos^2\theta$

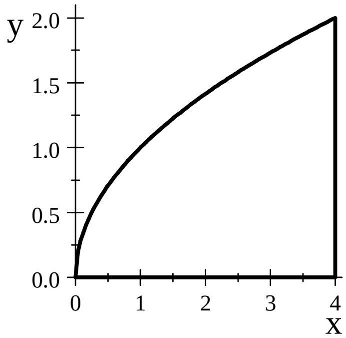
$$\oint_{\partial P} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\,dt = \int_0^{2\pi} 54\cos^2\theta\,dt = \int_0^{2\pi} 27(1+\cos 2\theta)\,dt = 27\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 54\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (20 points) For each integral, plot the region of integration and then compute the integral.

a. $I = \int_0^2 \int_{y^2}^4 \cos(x^{3/2})\, dx\, dy$

Solution:



$$I = \int_0^4 \int_0^{x^{1/2}} \cos(x^{3/2})\, dy\, dx = \int_0^4 \Big[\cos(x^{3/2})\, y \Big]_{y=0}^{x^{1/2}} dx$$

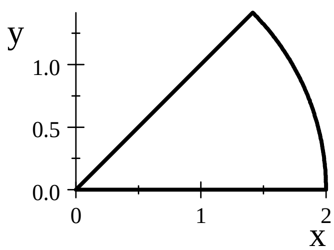
$$= \int_0^4 \cos(x^{3/2})\, x^{1/2}\, dx$$

$$u = x^{3/2} \qquad du = \frac{3}{2} x^{1/2}\, dx \qquad x^{1/2}\, dx = \frac{2}{3}\, du$$

$$I = \frac{2}{3} \int_0^8 \cos(u) du = \frac{2}{3} \Big[\sin(u) \Big]_0^8 = \frac{2}{3} (\sin(8))$$

b. $J = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \arctan\left(\frac{y}{x} \right) dx\, dy$

Solution:

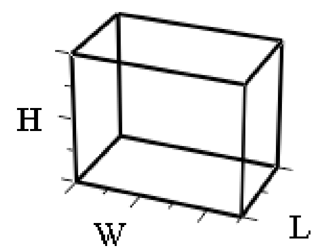


$$J = \int_0^{\pi/4} \int_0^2 \arctan\left(\frac{r \sin\theta}{r \cos\theta} \right) r\, dr\, d\theta$$

$$= \int_0^{\pi/4} \int_0^2 \theta\, r\, dr\, d\theta$$

$$= \left[\frac{\theta^2}{2} \right]_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^2 = \frac{\pi^2}{16}$$

14. (10 points) An aquarium in the shape of a rectangular solid has a slate bottom costing $\$6$ per ft^2, a glass front costing $\$2$ per ft^2, and aluminum sides and back costing $\$1$ per ft^2. There is no top. Let L be the length front to back, W be the width side to side and H be the height. Write a formula for the total cost, C, and find the dimensions and cost of the cheapest such aquarium if the volume is $V = 36$ ft^3. Do not use decimals.



Solution: The Cost is $C = 6LW + 2WH + 1(2LH + WH) = 6LW + 3WH + 2LH.$
The Volume constraint is $V = LWH = 36.$

Method 1: Lagrange Multuipliers:

$\vec{\nabla}C = (6W + 2H, 6L + 3H, 3W + 2L) \qquad \vec{\nabla}V = (WH, LH, LW)$

The Lagrange equations are:

$6W + 2H = \lambda WH \qquad 6L + 3H = \lambda LH \qquad 3W + 2L = \lambda LW$

Multiply the 1st equation by L, the 2nd equation by W, and the 3rd equation by H so the right sides are all λLWH.

Then equate them: $\lambda LWH = 6LW + 2LH = 6LW + 3WH = 3WH + 2LH$

The 1st and 2nd say $2LH = 3WH$ or $2L = 3W$. The 2nd and 3rd say $6LW = 2LH$ or $3W = H$.

Substitute $L = \frac{3}{2}W$ and $H = 3W$ into the volume constraint:

$\left(\frac{3}{2}W\right)(W)(3W) = 36 \qquad \frac{9}{2}W^3 = 36 \qquad W^3 = 8 \qquad W = 2$

Substituting back: $L = 3 \qquad H = 6$ and $C = 6 \cdot 3 \cdot 2 + 3 \cdot 2 \cdot 6 + 2 \cdot 3 \cdot 6 = \$108$

Method 3: Eliminate a Variable:

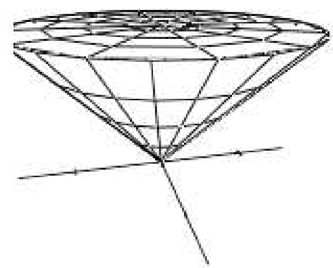
$H = \frac{36}{LW} \qquad C = 6LW + \frac{108}{L} + \frac{72}{W}$

$C_L = 6W - \frac{108}{L^2} = 0 \qquad C_W = 6L - \frac{72}{W^2} = 0 \qquad W = \frac{18}{L^2} \qquad L = \frac{12}{W^2}$

$W = \frac{18}{\left(\frac{12}{W^2}\right)^2} = 18\frac{W^4}{144} = \frac{1}{8}W^4 \qquad W^3 = 8 \qquad W = 2$

Substituting back: $L = \frac{12}{2^2} = 3 \qquad H = \frac{36}{3 \cdot 2} = 6$ and $C = 6 \cdot 3 \cdot 2 + \frac{108}{3} + \frac{72}{2} = \$108$

15. (25 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iint_{\partial V} \vec{F}\cdot d\vec{S}$



for the vector field $\vec{F} = (yz^2, xz^2, (x^2+y^2)z)$ and the solid cone $\sqrt{x^2+y^2} \le z \le 4$.

Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence:

$\vec{\nabla}\cdot\vec{F} = 0 + 0 + (x^2+y^2) = x^2+y^2$

b. Name your coordinate system: cylindrical

c. Express the divergence and the volume element in those coordinates:

$\vec{\nabla}\cdot\vec{F} = r^2$ $\qquad dV = r\,dr\,d\theta\,dz$

d. Compute the left hand side:

$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^4\int_r^4 (r^2)\,r\,dz\,dr\,d\theta = 2\pi\int_0^4 \Big[r^3 z\Big]_{z=r}^4 dr$$
$$= 2\pi\int_0^4 (4r^3 - r^4)\,dr = 2\pi\left[r^4 - \frac{r^5}{5}\right]_0^4 = 2\pi 4^4\left(1-\frac{4}{5}\right) = \frac{512\pi}{5}$$

Second the Right Hand Side:

The boundary surface consists of a cone C and a disk D with appropriate orientations.

e. Parametrize the disk D:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4)$

f. Compute the tangent vectors:

$\vec{e}_r = \big(\ \cos\theta,\quad \sin\theta,\quad 0\big)$

$\vec{e}_\theta = (-r\sin\theta,\quad r\cos\theta,\quad 0)$

g. Compute the normal vector:

$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (0,0,r)$ This is up. Oriented correctly.

h. Evaluate $\vec{F} = (yz^2, xz^2, (x^2+y^2)z)$ on the disk:

$\vec{F}\Big|_{\vec{R}(r,\theta)} = (16r\sin\theta, 16r\cos\theta, 4r^2)$

i. Compute the dot product:

$\vec{F}\cdot\vec{N} = 4r^3$

j. Compute the flux through D:

$$\iint_D \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^4 4r^3\,dr\,d\theta = 2\pi\Big[r^4\Big]_0^4 = 2\pi 4^4 = 512\pi$$

The cone C may be parametrized by:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

k. Compute the tangent vectors:

$\vec{e}_r = (\quad \cos\theta, \qquad \sin\theta, \qquad 1)$

$\vec{e}_\theta = (-r\sin\theta, \qquad r\cos\theta, \qquad 0)$

l. Compute the normal vector:

$\vec{N} = \hat{\imath}(0 - r\cos\theta) - \hat{\jmath}(0 - -r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta)$

$= (-r\cos\theta, -r\sin\theta, r)$ This is oriented up and in. We need down and out.

Reverse $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

m. Evaluate $\vec{F} = (yz^2, xz^2, (x^2 + y^2)z)$ on the cone:

$\vec{F}\Big|_{\vec{R}(r,\theta)} = (r^3\sin\theta, r^3\cos\theta, r^3)$

n. Compute the dot product:

$\vec{F} \cdot \vec{N} = r^4\cos\theta\sin\theta + r^4\sin\theta\cos\theta - r^4 = 2r^4\cos\theta\sin\theta - r^4 = r^4\sin2\theta - r^4$

o. Compute the flux through C:

$$\iint_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^4 r^4\sin2\theta - r^4\,dr\,d\theta = \left[-\frac{\cos2\theta}{2} - \theta\right]_0^{2\pi}\left[\frac{r^5}{5}\right]_0^4 = -2\pi\frac{4^5}{5} = -\frac{2048\pi}{5}$$

p. Compute the **TOTAL** right hand side:

$$\iint_{\partial V} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} + \iint_C \vec{F} \cdot d\vec{S} = 512\pi - \frac{2048\pi}{5} = \frac{512}{5}\pi$$ which agrees with (d).