1-12	/60
13	/16
14	/12
15	/12
Total	/100

Name________________________

MATH 251 Exam 1C Fall 2015

Sections 511/512 (circle one) Solutions P. Yasskin

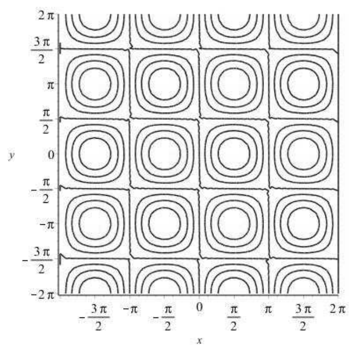
Multiple Choice: (5 points each. No part credit.)

1. If $\vec{a} = (2,-6,-2)$ and $\vec{b} = (-1,-2,1)$, then $|\vec{a} - 2\vec{b}| =$

a. 1
b. 3
c. 5
d. 6 Correct
e. 13

Solution: $|\vec{a} - 2\vec{b}| = |(4,-2,-4)| = \sqrt{16+4+16} = 6$

2. The plot at the right is the contour plot of which function?
HINT: Where is the level set with value 0?



a. $\sin(x)\cos(y)$ Correct
b. $\sin(x)\sin(y)$
c. $\cos(x)\cos(y)$
d. $\cos(x)\sin(y)$
e. $\sin(xy)$

Solution: There is a level set at $x = 0$ but not $x = \frac{\pi}{2}$. So it must have a $\sin(x)$ but not $\cos(x)$ factor. There is a level set at $y = \frac{\pi}{2}$ but not $y = 0$. So it must have a $\cos(y)$ but not $\sin(y)$ or $\sin(xy)$ factor.

3. Find the projection of the vector $\vec{u} = (1,1,3)$ onto the vector $\vec{v} = (2,1,-2)$ and is the angle between these vectors acute or obtuse?

a. $\left(\frac{-1}{11}, \frac{-1}{11}, \frac{-3}{11}\right)$, obtuse

b. $\left(\frac{-1}{11}, \frac{-1}{11}, \frac{-3}{11}\right)$, acute

c. $\left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$, obtuse Correct

d. $\left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$, acute

e. $\left(\frac{-2}{11}, \frac{-1}{11}, \frac{2}{11}\right)$, obtuse

Solution: $\vec{u} \cdot \vec{v} = 2 + 1 - 6 = -3 < 0$ So the angle is obtuse.

$$proj_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \frac{-3}{9}(2,1,-2) = \left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$$

4. Find the asymptotes of the hyperbola $4(x-3)^2 - 9(y-2)^2 = 36$.

a. $y = 2 \pm \frac{3}{2}(x-3)$

b. $y = 2 \pm \frac{2}{3}(x-3)$ Correct

c. $y = 3 \pm \frac{3}{2}(x-2)$

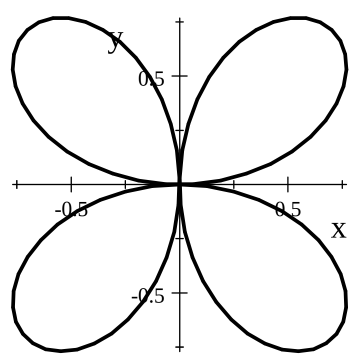
d. $y = 3 \pm \frac{2}{3}(x-2)$

e. $y = -3 \pm \frac{2}{3}(x+2)$

Solution: In standard form the hyperbola is $\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1$. The asymptotes are the cross $\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 0$. We solve:

$$\frac{(y-2)^2}{4} = \frac{(x-3)^2}{9} \qquad (y-2)^2 = \frac{4}{9}(x-3)^2 \qquad y-2 = \pm\frac{2}{3}(x-3) \qquad y = 2 \pm \frac{2}{3}(x-3)$$

5. The plot at the right is the graph of which polar curve?



a. $r = \cos(2\theta)$

b. $r = \sin(2\theta)$ Correct

c. $r = \cos(4\theta)$

d. $r = \sin(4\theta)$

Solution: $\sin(4\theta)$ and $\cos(4\theta)$ would have 8 loops. $r = 0$ at $\theta = 0$ which is equation (b).

6. Find the volume of the parallelepiped with edge vectors $\vec{u} = (1,-1,1)$, $\vec{v} = (2,1,0)$, and $\vec{w} = (0,1,-2)$.

- **a.** -8
- **b.** -4
- **c.** 4 Correct
- **d.** 6
- **e.** 8

Solution: $V = |\vec{u} \times \vec{v} \cdot \vec{w}| = \left| \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{vmatrix} \right| = |-4| = 4$

7. Find the line through the points $A = (8,4,-6)$ and $B = (10,5,-9)$. It passes through the xy-plane at:

- **a.** $(4,2,0)$ Correct
- **b.** $(8,4,0)$
- **c.** $(10,5,0)$
- **d.** $(-8,-4,0)$
- **e.** $(-10,-5,0)$

Solution: $\vec{v} = \overrightarrow{AB} = (2,1,-3)$ $\quad X = A + t\vec{v}$
$(x,y,z) = (8,4,-6) + t(2,1,-3) = (8+2t, 4+t, -6-3t)$
$z = -6 - 3t = 0$ at $t = -2$ So $(x,y,z) = (8,4,-6) - 2(2,1,-3) = (4,2,0)$

8. Compute $\lim\limits_{h \to 0} \dfrac{(4x + 4h + 3y)^2 - (4x + 3y)^2}{h}$

- **a.** $4x + 3y$
- **b.** $8x + 6y$
- **c.** $16x + 12y$
- **d.** $24x + 18y$
- **e.** $32x + 24y$ Correct

Solution: $\dfrac{\partial}{\partial x}(4x + 3y)^2 = 2(4x + 3y)4 = 32x + 24y$

9. Find the plane tangent to the graph of $z = x^3 e^{2y}$ at $(2,0)$. The z-intercept is

a. -24
b. -16 Correct
c. -8
d. 8
e. 16

Solution:

$f(x,y) = x^3 e^{2y}$ $\quad f(2,0) = 8 \quad z = f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0)$

$f_x(x,y) = 3x^2 e^{2y}$ $\quad f_x(2,0) = 12 \quad = 8 + 12(x-2) + 16(y)$

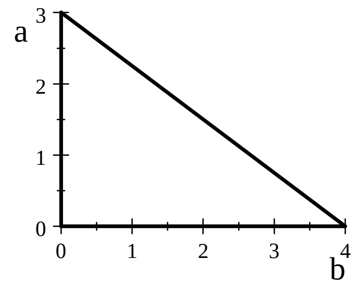
$f_y(x,y) = 2x^3 e^{2y}$ $\quad f_y(2,0) = 16 \quad = 12x + 16y - 16 \quad c = -16$

10. If $Q(2,3) = 6$ and $\dfrac{\partial Q}{\partial x}(2,3) = 0.3$ and $\dfrac{\partial Q}{\partial y}(2,3) = -0.2$, estimate $Q(2.2,2.7)$.

a. 5.88
b. 5.9
c. 6.0
d. 6.1
e. 6.12 Correct

Solution: The linear approximation says:
$Q(x,y) \approx Q_{\tan}(x,y) = Q(a,b) + Q_x(a,b)(x-a) + Q_y(a,b)(y-b)$
Here $(x,y) = (2.2,2.7)$ and $(a,b) = (2,3)$. So
$Q(2.2,2.7) \approx Q(2,3) + Q_x(2,3)(.2) + Q_y(2,3)(-.3) = 6 + 0.3(.2) - 0.2(-.3) = 6.12$

11. A right triangle has sides a and b. If a increases from 3 cm to 3.02 cm, while b decreases from 4 cm to 3.98 cm, use the linear approximation to determine whether the hypotenuse increases or decreases and by how much.



- **a**. increases by 0.0028
- **b**. increases by 0.004
- **c**. increases by 0.028
- **d**. decreases by 0.004 Correct
- **e**. decreases by 0.028

Solution: $c = \sqrt{a^2 + b^2}$

$$\Delta c \approx dc = \frac{\partial c}{\partial a} da + \frac{\partial c}{\partial b} db = \frac{a}{\sqrt{a^2+b^2}} da + \frac{b}{\sqrt{a^2+b^2}} db$$

$$= \left(\frac{3}{5}\right)(.02) + \left(\frac{4}{5}\right)(-.02) = -0.004 \quad \text{decreases}$$

12. The oxygen density in a fish tank is given by $\rho = (x^2 + y^2)(15 - z)$. Currently, a fish is at $\vec{r} = (3,4,5)$ and has velocity $\vec{v} = (3,2,1)$. What is the rate of change of the oxygen density as seen by the fish?

- **a**. 9
- **b**. 115
- **c**. 315 Correct
- **d**. 365
- **e**. 375

Solution: $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt} = (2x(15-z))v_1 + (2y(15-z))v_2 + ((x^2+y^2)(-1))v_3$$

$$= (2 \cdot 3(15-5))3 + (2 \cdot 4(15-5))2 + ((3^2+4^2)(-1))1 = 315$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. $\left(16 \text{ points}\right)$ For the parametric curve $\vec{r}(t) = \left(\frac{1}{2}t^2, \frac{2}{3}t^3, \frac{1}{2}t^4\right)$ compute each of the following:

a. velocity $\vec{v}$

Solution: $\vec{v} = (t, 2t^2, 2t^3)$

b. speed $|\vec{v}|$ HINT: The quantity inside the square root is a perfect square.

Solution: $|\vec{v}| = \sqrt{t^2 + 4t^4 + 4t^6} = t + 2t^3$

c. arc length $L = \int_{(0,0,0)}^{\left(2,\frac{16}{3},8\right)} ds$

Solution: $L = \int_0^2 |\vec{v}|\, dt = \int_0^2 (t + 2t^3)\, dt = \left[\frac{t^2}{2} + \frac{t^4}{2}\right]_0^2 = 2 + 8 = 10$

d. acceleration $\vec{a}$

Solution: $\vec{a} = (1, 4t, 6t^2)$

e. unit binormal $\hat{B}$

Solution: $\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t & 2t^2 & 2t^3 \\ 1 & 4t & 6t^2 \end{vmatrix} = \hat{\imath}(12t^4 - 8t^4) - \hat{\jmath}(6t^3 - 2t^3) + \hat{k}(4t^2 - 2t^2) = (4t^4, -4t^3, 2t^2)$

$|\vec{v} \times \vec{a}| = \sqrt{16t^8 + 16t^6 + 4t^4} = 4t^4 + 2t^2 = 2t^2(2t^2 + 1)$

$\hat{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{1}{2t^2(2t^2+1)}(4t^4, -4t^3, 2t^2) = \left(\frac{2t^2}{2t^2+1}, \frac{-2t}{2t^2+1}, \frac{1}{2t^2+1}\right)$

f. tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(t + 2t^3) = 1 + 6t^2$

14. $\left(12 \text{ points}\right)$ A wire has the shape of the parametric curve $\vec{r}(t) = \left(\frac{1}{2}t^2, \frac{2}{3}t^3, \frac{1}{2}t^4\right)$ between $(0,0,0)$ and $\left(2, \frac{16}{3}, 8\right)$. Find the mass of the wire if the linear mass density is $\rho = 4xz$. Don't simplify the answer.

Solution: $\vec{v} = (t, 2t^2, 2t^3)$ $\quad |\vec{v}| = t + 2t^3$ $\quad \rho = 4xz = 4\left(\frac{1}{2}t^2\right)\left(\frac{1}{2}t^4\right) = t^6$

$M = \int_{(0,0,0)}^{\left(2,\frac{16}{3},8\right)} \rho\, ds = \int_0^2 4xz|\vec{v}|\, dt = \int_0^2 t^6(t + 2t^3)\, dt = \int_0^2 (t^7 + 2t^9)\, dt = \left[\frac{t^8}{8} + \frac{t^{10}}{5}\right]_0^2$

$= 2^5 + \frac{2^{10}}{5} = \frac{1184}{5}$

15. $\left(12 \text{ points}\right)$ A mass slides along a wire which has the shape of the parametric curve $\vec{r}(t) = \left(\frac{1}{2}t^2, \frac{2}{3}t^3, \frac{1}{2}t^4\right)$ between $(0,0,0)$ and $\left(2, \frac{16}{3}, 8\right)$ under the action of the force $\vec{F} = (4z, 3y, 2x)$. Find the work done by the force.

Solution: $\vec{F} = (4z, 3y, 2x) = (2t^4, 2t^3, t^2)$ $\quad \vec{v} = (t, 2t^2, 2t^3)$

$\vec{F} \cdot \vec{v} = 2t^4 t + 2t^3 2t^2 + t^2 2t^3 = 8t^5$

$W = \int_{(0,0,0)}^{\left(2,\frac{16}{3},8\right)} \vec{F} \cdot d\vec{s} = \int_0^2 \vec{F} \cdot \vec{v}\, dt = \int_0^2 8t^5\, dt = \left[\frac{4t^6}{3}\right]_0^2 = \frac{2^8}{3} = \frac{256}{3}$