Name________________________

1-9	/45
10	/30
11	/15
12	/15
Total	/105

MATH 251 Exam 2A Fall 2015

Sections 511/512 (circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the volume below the surface $z = x + 2y$ above the region in the xy-plane between $y = x^2$ and $y = 3x$.

a. $\frac{-71}{20}$

b. $\frac{71}{20}$

c. $\frac{972}{5}$

d. $\frac{783}{20}$ Correct

e. $\frac{1944}{5}$

Solution: Find the intersections: $x^2 = 3x \quad \Rightarrow \quad x = 0, 3$

$$V = \int_0^3 \int_{x^2}^{3x} (x + 2y)\, dy\, dx = \int_0^3 \Big[xy + y^2 \Big]_{y=x^2}^{3x} dx = \int_0^3 (3x^2 + 9x^2) - (x^3 + x^4)\, dx$$

$$= \left[12\frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \right]_0^3 = \left(12 \cdot 9 - \frac{81}{4} - \frac{243}{5} \right) = \frac{783}{20}$$

2. The temperature on a hot plate with dimensions $-4\pi \le x \le 4\pi$ and $0 \le y \le 6$ is $T = x^2 y + \cos^2 x$. Find the average temperature.

a. 24π

b. $768\pi^3$

c. $768\pi^3 + 24\pi$

d. $16\pi^2 + \frac{1}{2}$ Correct

e. $16\pi^2$

Solution: $A = \int_{-4\pi}^{4\pi} \int_0^6 1\, dy\, dx = 6 \cdot 8\pi = 48\pi$

$$\iint T\, dA = \int_{-4\pi}^{4\pi} \int_0^6 (x^2 y + \cos^2 x)\, dy\, dx = \int_{-4\pi}^{4\pi} \left[\frac{x^2 y^2}{2} + y\cos^2 x \right]_{y=0}^{6} dx$$

$$= \int_{-4\pi}^{4\pi} \left(18x^2 + 6\frac{1 + \cos(2x)}{2} \right) dx = \left[6x^3 + 3\left(x + \frac{\sin(2x)}{2} \right) \right]_{-4\pi}^{4\pi}$$

$$= 2(6 \cdot 4^3 \pi^3 + 3(4\pi + 0)) = 768\pi^3 + 24\pi$$

$$T_{ave} = \frac{1}{A} \iint T\, dA = \frac{768\pi^3 + 24\pi}{48\pi} = 16\pi^2 + \frac{1}{2}$$

3. Find the centroid of the region above $y = 2x^2$ below $y = 18$.

a. $\left(0, \frac{27}{5}\right)$

b. $\left(0, \frac{36}{5}\right)$

c. $\left(0, \frac{54}{5}\right)$ Correct

d. $\left(0, \frac{1944}{5}\right)$

e. $\left(0, \frac{3888}{5}\right)$

Solution: $2x^2 = 18 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$

$$A = \iint 1\,dA = \int_{-3}^{3}\int_{2x^2}^{18} 1\,dy\,dx = \int_{-3}^{3}(18 - 2x^2)\,dx = \left[18x - 2\frac{x^3}{3}\right]_{-3}^{3} = 2\left(18 \cdot 3 - 2\frac{3^3}{3}\right) = 72$$

By symmetry, $\bar{x} = 0$.

$$A_y = \iint y\,dA = \int_{-3}^{3}\int_{2x^2}^{18} y\,dy\,dx = \int_{-3}^{3}\left[\frac{y^2}{2}\right]_{2x^2}^{18} dx = \frac{1}{2}\int_{-3}^{3}(18^2 - 4x^4)\,dx = \frac{1}{2}\left[18^2 x - 4\frac{x^5}{5}\right]_{-3}^{3}$$

$$= \left(18^2 \cdot 3 - 4\frac{3^5}{5}\right) = 3^5\left(4 - \frac{4}{5}\right) = \frac{16}{5}3^5$$

$$\bar{y} = \frac{A_y}{A} = \frac{16 \cdot 3^5}{5 \cdot 72} = \frac{54}{5}$$

4. Find all critical points of the function $f(x,y) = 4x^2 + 9y^2 + \frac{432}{xy}$. Select from:

$A = (2,3) \quad B = (-2,3) \quad C = (2,-3) \quad D = (-2,-3)$

$E = (3,2) \quad F = (-3,2) \quad G = (3,-2) \quad H = (-3,-2)$

Note $432 = 2^4 3^3$

a. A,B,C,D

b. E,F,G,H

c. A,D

d. B,C

e. E,H Correct

Solution:

$$f_x = \frac{d}{dx}\left(4x^2 + 9y^2 + \frac{432}{xy}\right) = \frac{8}{x^2 y}(x^3 y - 54) = 0 \qquad x^3 y = 54$$

$$f_y = \frac{d}{dy}\left(4x^2 + 9y^2 + \frac{432}{xy}\right) = \frac{18}{xy^2}(xy^3 - 24) = 0 \qquad xy^3 = 24$$

Multiply: $x^4 y^4 = 54 \cdot 24 = 2^4 3^4 \quad \Rightarrow \quad xy = \pm 6$

Divide: $\frac{x^2}{y^2} = \frac{54}{24} = \frac{9}{4} \quad \Rightarrow \quad \frac{x}{y} = \pm\frac{3}{2}$

Multiply results: $xy\frac{x}{y} = x^2 = \pm 6 \cdot \frac{3}{2} = \pm 9$ Must have $x^2 = 9 \quad x = \pm 3$

Divide results: $xy\frac{y}{x} = y^2 = \pm 6 \cdot \frac{2}{3} = \pm 4$ Must have $y^2 = 4 \quad y = \pm 2$

Looking at $x^3 y = 54$, x and y must be both positive or both negative. So $(3,2), (-3,-2)$

5. Select all of the following statements which are consistent with this countour plot?

A. There is a local maximum at $(1,1)$.

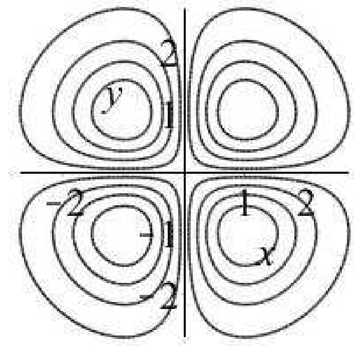
B. There is a local minimum at $(1,1)$.

C. There is a saddle point at $(1,1)$.

D. There is a local maximum at $(0,0)$.

E. There is a local minimum at $(0,0)$.

F. There is a saddle point at $(0,0)$.



a. A,B,D,E
b. C,F
c. C,D,E
d. A,B,F   Correct

Solution: A,B but not D,E because there should be circles around a local maximum or minimum.

6. The function $f = \frac{4}{x} - \frac{2}{y} - xy$ has a critical point at $(x,y) = (2,-1)$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum   Correct
b. Local Maximum
c. Inflection Point
d. Saddle Point
e. Test Fails

Solution: $f_x = -\frac{4}{x^2} - y \qquad f_y = \frac{2}{y^2} - x$

$f_{xx} = \frac{8}{x^3} \qquad f_{yy} = -\frac{4}{y^3} \qquad f_{xy} = -1$

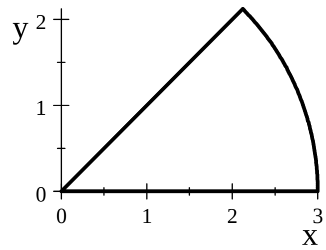
$f_{xx}(2,-1) = 1 \qquad f_{yy}(2,-1) = 4 \qquad f_{xy}(2,-1) = -1 \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3$

Local Minimum

7. Find the mass of the $\frac{1}{8}$ of the circle

$$x^2 + y^2 \le 9 \quad \text{for} \quad 0 \le y \le x$$

if the surface density is $\delta = x$.



a. 9

b. $\frac{9}{2}$

c. $9\sqrt{2}$

d. $\frac{9}{\sqrt{2}}$ Correct

e. $\frac{9}{2\sqrt{2}}$

Solution: In polar coordinates, the density is $\delta = x = r\cos\theta$.

$$M = \iint \delta\, dA = \int_0^{\pi/4} \int_0^3 r\cos\theta\, r\, dr\, d\theta = \left[\frac{r^3}{3}\right]_0^3 \left[\sin\theta\right]_0^{\pi/4} = 9\left(\frac{1}{\sqrt{2}}\right) = \frac{9}{\sqrt{2}}$$

8. Find the y-component of the center of mass of the $\frac{1}{8}$ of the circle $x^2 + y^2 \le 9$ for $0 \le y \le x$ if the surface density is $\delta = x$.

a. $\frac{9\sqrt{2}}{16}$ Correct

b. $\frac{9\sqrt{2}}{8}$

c. $\frac{81}{16}$

d. $\frac{81}{32}$

e. $\frac{1}{\sqrt{2}}$

Solution: In polar coordinates, the density is $\delta = x = r\cos\theta$ and $y = r\sin\theta$.

$$M_y = \iint y\delta\, dA = \int_0^{\pi/4} \int_0^3 r\sin\theta\, r\cos\theta\, r\, dr\, d\theta = \left[\frac{r^4}{4}\right]_0^3 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{81}{4}\left(\frac{1}{4}\right) = \frac{81}{16}$$

$$\bar{y} = \frac{M_y}{M} = \frac{81}{16}\,\frac{\sqrt{2}}{9} = \frac{9\sqrt{2}}{16}$$

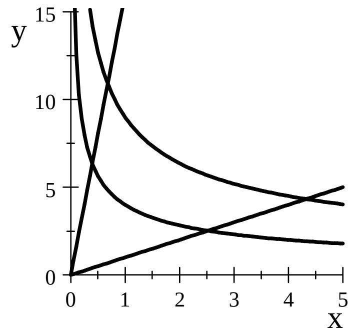
9. Compute $\iint y\,dA$ over the "diamond" shaped region in the first quadrant bounded by

$xy^2 = 8 \qquad xy^2 = 27 \qquad y = x \qquad y = 8x$

HINTS: Use the coordinates

$x = \dfrac{u}{v^2} \qquad y = uv$

Find the boundaries and Jacobian.



a. $19\ln 2$ Correct

b. $80\ln 2$

c. $15\ln 3$

d. $65\ln 2$

e. $65\ln 3$

Solution: Let $(x,y) = \left(\dfrac{u}{v^2}, uv\right)$

Boundaries: $xy^2 = \dfrac{u}{v^2}u^2v^2 = u^3 = 8,27 \qquad \Rightarrow \qquad u = 2,3$

$\dfrac{y}{x} = uv\dfrac{v^2}{u} = v^3 = 1,8 \qquad \Rightarrow \qquad v = 1,2$

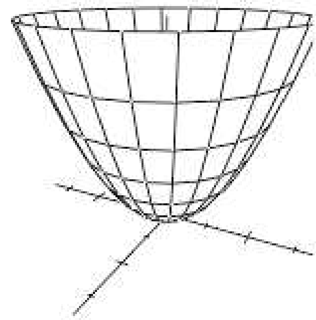
Jacobian: $J = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} \dfrac{1}{v^2} & v \\ -\dfrac{2u}{v^3} & u \end{matrix}\right\| = \left|\dfrac{u}{v^2} - -2\dfrac{u}{v^2}\right| = 3\dfrac{u}{v^2}$

Evaluate the integral:

$$\iint y\,dA = \int\int yJ\,du\,dv = \int_1^2\int_2^3 uv3\frac{u}{v^2}\,du\,dv = [u^3]_2^3[\ln v]_1^2 = (27-8)\ln 2 = 19\ln 2$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (30 points) Consider the piece of the paraboloid surface $z = 2x^2 + 2y^2$ for $z \le 18$.



- Find the mass of the paraboloid if the surface mass density is $\delta = \dfrac{3z}{x^2 + y^2}$.
- Find the flux of the electric field $\vec{E} = \left(\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}, 0 \right)$ down and out of the paraboloid.

Parametrize the surface as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2r^2)$ and follow these steps:

a. Find the coordinate tangent vectors:

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta & \sin\theta & 4r \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 0 \end{array}$$

b. Find the normal vector and check its orientation.

$\vec{N} = i(-4r^2\cos\theta) - j(4r^2\sin\theta) + k(r\cos^2\theta - -r\sin^2\theta) = (-4r^2\cos\theta, -4r^2\sin\theta, r)$

This is up and in. We need down and out. So reverse the normal.

$\vec{N} = (4r^2\cos\theta, 4r^2\sin\theta, -r)$

c. Find the length of the normal vector.

$\left|\vec{N}\right| = \sqrt{16r^4\cos^2\theta + 16r^4\sin^2\theta + r^2} = \sqrt{16r^4 + r^2} = r\sqrt{16r^2 + 1}$

d. Evaluate the density $\delta = \dfrac{3z}{x^2 + y^2}$ on the paraboloid.

$$\delta\left(\vec{R}(r,\theta)\right) = \frac{3 \cdot 2r^2}{r^2} = 6$$

e. Compute the mass.

Find the limit on r: $\quad z = 2r^2 = 18 \qquad \Rightarrow \qquad r = 3$

$$M = \iint \delta\, dS = \int_0^{2\pi} \int_0^3 6\, r\sqrt{16r^2 + 1}\; dr\, d\theta = 2\pi \cdot 6\left[\frac{2}{3 \cdot 32}(16r^2 + 1)^{3/2}\right]_0^3$$
$$= \frac{\pi}{4}(145^{3/2} - 1)$$

f. Evaluate the electric field $\vec{E} = \left(\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}, 0\right)$ on the paraboloid.

$$\vec{E}\left(\vec{R}(r,\theta)\right) = \left(\frac{r\cos\theta}{r^2}, \frac{r\sin\theta}{r^2}, 0\right) = \left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}, 0\right)$$

g. Compute the flux.

$$\iint \vec{E} \cdot d\vec{S} = \int_0^{2\pi} \int_0^3 \vec{E} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^3 \left(\frac{\cos\theta}{r} 4r^2 \cos\theta + \frac{\sin\theta}{r} 4r^2 \sin\theta\right) dr\, d\theta$$
$$= 2\pi \int_0^3 (4r)\, dr = 8\pi\left[\frac{r^2}{2}\right]_0^3 = 36\pi$$

11. (15 points) Find the volume of the largest rectangular solid which sits on the xy-plane and has its upper 4 vertices on the paraboloid $z + x^2 + 4y^2 = 64$.

Solution: Minimize $V = 4xyz$ subject to the constraint $g = z + x^2 + 4y^2 = 64$.
Use Lagrange Multipliers

$\vec{\nabla}V = (4yz, 4xz, 4xy)$ $\quad \vec{\nabla}g = (2x, 8y, 1)$ $\quad \vec{\nabla}f = \lambda\vec{\nabla}g$ $\quad 4yz = \lambda 2x$ $\quad 4xz = \lambda 8y$ $\quad 4xy = \lambda$

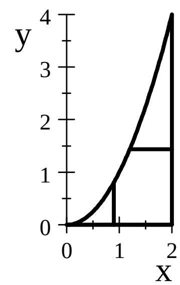
$\lambda = \dfrac{4yz}{2x} = \dfrac{4xz}{8y} = 4xy$ or $\dfrac{z}{2x} = x$ and $\dfrac{z}{8y} = y$ or $x^2 = \dfrac{z}{2}$ and $y^2 = \dfrac{z}{8}$

$g = z + x^2 + 4y^2 = z + \dfrac{z}{2} + \dfrac{z}{2} = 2z = 64 \quad \Rightarrow \quad z = 32 \quad x^2 = 16 \quad x = 4 \quad y^2 = 4 \quad y = 2$

$V = 4xyz = 4 \cdot 4 \cdot 2 \cdot 32 = 1024$

12. (15 points) Draw the region of integration and compute $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3 + 1}\, dx\, dy$

Solution: To reverse the order of integration plot the region $0 \le y \le 4$, $\sqrt{y} \le x \le 2$.
Include a horizontal line to indicate the x limits.
Add a vertical line to indicate the new y limits.
Write the new integral and compute it.



$$\int_0^2 \int_0^{x^2} \sqrt{x^3 + 1}\, dy\, dx = \int_0^2 \sqrt{x^3 + 1}\, [y]_0^{x^2}\, dx = \int_0^2 \sqrt{x^3 + 1}\, x^2\, dx = \frac{2}{9}(x^3 + 1)^{3/2}\Big|_0^2 = \frac{2}{9}(9^{3/2} - 1) = \frac{52}{9}$$