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MATH 251 Exam 2C Fall 2015

Sections 511/512 (circle one) Solutions P. Yasskin

1-9	/45
10	/30
11	/15
12	/15
Total	/105

Multiple Choice: (5 points each. No part credit.)

1. Find the volume below the surface $z = x + 3y$ above the region in the xy-plane between $y = x^2$ and $y = 2x$.

a. -32

b. 32

c. $\frac{32}{15}$

d. $\frac{92}{15}$

e. $\frac{116}{15}$ Correct

Solution: Find the intersections: $x^2 = 2x \quad \Rightarrow \quad x = 0, 2$

$$V = \int_0^2 \int_{x^2}^{2x} (x + 3y)\, dy\, dx = \int_0^2 \left[xy + 3\frac{y^2}{2} \right]_{y=x^2}^{2x} dx = \int_0^2 (2x^2 + 6x^2) - \left(x^3 + \frac{3}{2} x^4 \right) dx$$

$$= \left[8\frac{x^3}{3} - \frac{x^4}{4} - \frac{3}{2}\frac{x^5}{5} \right]_0^2 = \left(8 \cdot \frac{8}{3} - 4 - 3 \cdot \frac{16}{5} \right) = \frac{116}{15}$$

2. The temperature on a triangular hot plate with vertices at $(0,0), (0,4)$ and $(2,0)$ is $T = xy$. Find the average temperature.

a. $\frac{1}{3}$

b. $\frac{2}{3}$ Correct

c. 1

d. $\frac{4}{3}$

e. $\frac{8}{3}$

Solution: $A = \int_0^2 \int_0^{4-2x} 1\, dy\, dx = \int_0^2 (4 - 2x)\, dx = [4x - x^2]_0^2 = 4$

$$\iint T\, dA = \int_0^2 \int_0^{4-2x} xy\, dy\, dx = \int_0^2 \left[\frac{xy^2}{2} \right]_{y=0}^{4-2x} dx = \frac{1}{2} \int_0^2 x(4 - 2x)^2\, dx = \frac{1}{2} \int_0^2 (16x - 16x^2 + 4x^3)\, dx$$

$$= \frac{1}{2} \left[16\frac{x^2}{2} - 16\frac{x^3}{3} + x^4 \right]_0^2 = \frac{1}{2} \left(16\frac{2^2}{2} - 16\frac{2^3}{3} + 2^4 \right) = \frac{8}{3}$$

$$T_{ave} = \frac{1}{A} \iint T\, dA = \frac{8}{3}\frac{1}{4} = \frac{2}{3}$$

3. Find the centroid of the region above $y = 5x^2$ below $y = 20$.

- **a.** $(0,8)$
- **b.** $(0,10)$
- **c.** $(0,12)$ Correct
- **d.** $(0,15)$
- **e.** $(0,16)$

Solution: $5x^2 = 20 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

$$A = \iint 1\,dA = \int_{-2}^{2}\int_{5x^2}^{20} 1\,dy\,dx = \int_{-2}^{2}(20 - 5x^2)\,dx = \left[20x - 5\frac{x^3}{3}\right]_{-2}^{2} = 2\left(40 - \frac{40}{3}\right) = \frac{160}{3}$$

By symmetry, $\bar{x} = 0$.

$$A_y = \iint y\,dA = \int_{-2}^{2}\int_{5x^2}^{20} y\,dy\,dx = \int_{-2}^{2}\left[\frac{y^2}{2}\right]_{5x^2}^{20} dx = \frac{1}{2}\int_{-2}^{2}(400 - 25x^4)\,dx = \frac{1}{2}[400x - 5x^5]_{-2}^{2}$$

$$= 800 - 160 = 640$$

$$\bar{y} = \frac{A_y}{A} = \frac{640 \cdot 3}{160} = 12$$

4. Find all critical points of the function $f(x,y) = 4x^2 + 9y^2 + \frac{432}{xy}$. Select from:

$A = (3,2) \quad B = (-3,2) \quad C = (3,-2) \quad D = (-3,-2)$

$E = (2,3) \quad F = (-2,3) \quad G = (2,-3) \quad H = (-2,-3)$

Note $432 = 2^4 3^3$

- **a.** A,B,C,D
- **b.** E,F,G,H
- **c.** B,C
- **d.** A,D Correct
- **e.** E,H

Solution:

$$f_x = \frac{d}{dx}\left(4x^2 + 9y^2 + \frac{432}{xy}\right) = \frac{8}{x^2y}(x^3y - 54) = 0 \qquad x^3y = 54$$

$$f_y = \frac{d}{dy}\left(4x^2 + 9y^2 + \frac{432}{xy}\right) = \frac{18}{xy^2}(xy^3 - 24) = 0 \qquad xy^3 = 24$$

Multiply: $x^4y^4 = 54 \cdot 24 = 2^4 3^4 \quad \Rightarrow \quad xy = \pm 6$

Divide: $\frac{x^2}{y^2} = \frac{54}{24} = \frac{9}{4} \quad \Rightarrow \quad \frac{x}{y} = \pm\frac{3}{2}$

Multiply results: $xy\frac{x}{y} = x^2 = \pm 6 \cdot \frac{3}{2} = \pm 9$ Must have $x^2 = 9 \quad x = \pm 3$

Divide results: $xy\frac{y}{x} = y^2 = \pm 6 \cdot \frac{2}{3} = \pm 4$ Must have $y^2 = 4 \quad y = \pm 2$

Looking at $x^3y = 54$, x and y must be both positive or both negative. So $(3,2), (-3,-2)$

5. Select all of the following statements which are consistent with this countour plot?

A. There is a local maximum at $(0,0)$.

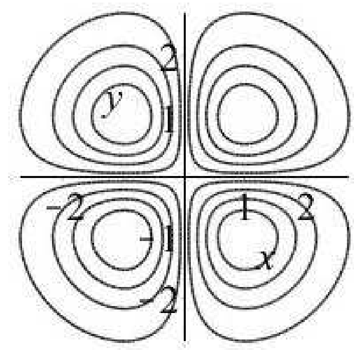
B. There is a local minimum at $(0,0)$.

C. There is a saddle point at $(0,0)$.

D. There is a local maximum at $(1,1)$.

E. There is a local minimum at $(1,1)$.

F. There is a saddle point at $(1,1)$.



a. A,B,D,E

b. C,F

c. C,D,E   Correct

d. A,B,F

Solution: D,E but not A,B because there should be circles around a local maximum or minimum.

6. The function $f = -\frac{4}{x} + \frac{2}{y} + xy$ has a critical point at $(x,y) = (2,-1)$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum

b. Local Maximum   Correct

c. Inflection Point

d. Saddle Point

e. Test Fails

Solution: $f_x = \frac{4}{x^2} + y \qquad f_y = -\frac{2}{y^2} + x$

$f_{xx} = -\frac{8}{x^3} \qquad f_{yy} = \frac{4}{y^3} \qquad f_{xy} = 1$

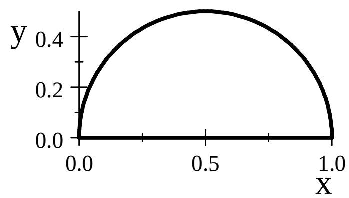
$f_{xx}(2,-1) = -1 \qquad f_{yy}(2,-1) = -4 \qquad f_{xy}(2,-1) = 1 \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3$

Local Maximum

7. Find the mass of the region in the upper half of the circle

$r = \cos\theta$

if the surface density is $\delta = y$.



a. $\frac{1}{12}$ Correct

b. $\frac{1}{9}$

c. $\frac{1}{6}$

d. $\frac{1}{3}$

e. $\frac{1}{2}$

Solution: The density is $\delta = y = r\sin\theta$.

$$M = \iint \delta\, dA = \int_0^{\pi/2}\int_0^{\cos\theta} r\sin\theta\, r\, dr\, d\theta = \int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{\cos\theta}\sin\theta\, d\theta = \frac{1}{3}\int_0^{\pi/2}\cos^3\theta\sin\theta\, d\theta$$

$$= \frac{1}{3}\left[\frac{-\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{1}{3}\left(0+\frac{1}{4}\right) = \frac{1}{12}$$

8. Find the x-component of the center of mass of the region in the upper half of the circle $r = \cos\theta$ if the surface density is $\delta = y$.

a. $\frac{1}{24}$

b. $\frac{1}{4}$

c. $\frac{1}{3}$

d. $\frac{1}{2}$ Correct

e. $\frac{2}{3}$

Solution: The density is $\delta = y = r\sin\theta$ and $x = r\cos\theta$.

$$M_x = \iint x\,\delta\, dA = \int_0^{\pi/2}\int_0^{\cos\theta} r\cos\theta\, r\sin\theta\, r\, dr\, d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{\cos\theta}\cos\theta\sin\theta\, d\theta = \frac{1}{4}\int_0^{\pi/2}\cos^5\theta\sin\theta\, d\theta$$

$$= \frac{1}{4}\left[\frac{-\cos^6\theta}{6}\right]_0^{\pi/2} = \frac{1}{4}\left(0+\frac{1}{6}\right) = \frac{1}{24} \qquad \bar{x} = \frac{M_x}{M} = \frac{1}{24}\,\frac{12}{1} = \frac{1}{2}$$

You can expect this by symmetry.

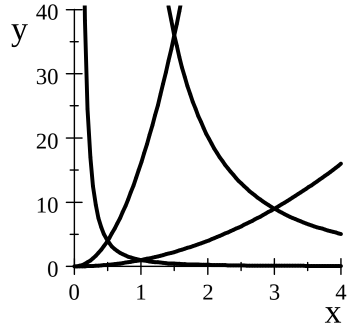
9. Compute $\iint x\,dA$ over the “diamond” shaped region in the first quadrant bounded by

$x^2y = 1 \qquad x^2y = 81 \qquad y = x^2 \qquad y = 16x^2$

HINTS: Use the coordinates

$x = \frac{u}{v} \qquad y = u^2v^2$

Find the boundaries and Jacobian.



a. $19\ln 2$

b. $80\ln 2$ Correct

c. $15\ln 3$

d. $65\ln 2$

e. $65\ln 3$

Solution: Let $(x,y) = (\frac{u}{v}, u^2v^2)$

Boundaries: $x^2y = (\frac{u}{v})^2u^2v^2 = u^4 = 1, 81 \qquad \Rightarrow \qquad u = 1, 3$

$$\frac{y}{x^2} = u^2v^2\frac{v^2}{u^2} = v^4 = 1, 16 \qquad \Rightarrow \qquad v = 1, 2$$

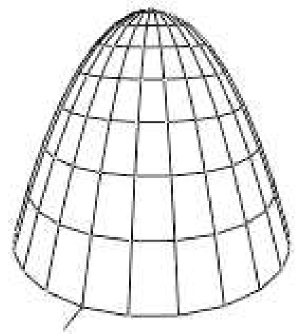
Jacobian: $J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} \frac{1}{v} & 2uv^2 \\ -\frac{u}{v^2} & 2u^2v \end{matrix}\right\| = |2u^2 - -2u^2| = 4u^2$

Evaluate the integral:

$$\iint x\,dA = \int\int x\,J\,du\,dv = \int_1^2\int_1^3 \frac{u}{v}\,4u^2\,du\,dv = [u^4]_1^3[\ln v]_1^2 = (81-1)\ln 2 = 80\ln 2$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (30 points) Consider the piece of the paraboloid surface $z = 18 - 2x^2 - 2y^2$ above the xy-plane.



- Find the mass of the paraboloid if the surface mass density is $\delta = z + 2x^2 + 2y^2$.
- Find the flux of the electric field $\vec{E} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, 0 \right)$ down into the paraboloid.

Parametrize the surface as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 18 - 2r^2)$ and follow these steps:

a. Find the coordinate tangent vectors:

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta & \sin\theta & -4r \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 0 \end{array}$$

b. Find the normal vector and check its orientation.

$\vec{N} = i(4r^2\cos\theta) - j(-4r^2\sin\theta) + k(r\cos^2\theta - -r\sin^2\theta) = (4r^2\cos\theta, 4r^2\sin\theta, r)$
This is up and out. We need down and in. So reverse the normal.

$\vec{N} = (-4r^2\cos\theta, -4r^2\sin\theta, -r)$

c. Find the length of the normal vector.

$\left|\vec{N}\right| = \sqrt{16r^4\cos^2\theta + 16r^4\sin^2\theta + r^2} = \sqrt{16r^4 + r^2} = r\sqrt{16r^2 + 1}$

d. Evaluate the density $\delta = z + 2x^2 + 2y^2$ on the paraboloid.

$$\delta\left(\vec{R}(r,\theta)\right) = (18 - 2r^2) + 2r^2 = 18$$

e. Compute the mass.

Find the limit on r: $z = 18 - 2r^2 = 0 \quad \Rightarrow \quad r = 3$

$$M = \iint \delta\, dS = \int_0^{2\pi}\int_0^3 18r\sqrt{16r^2+1}\; dr\, d\theta = 2\pi \cdot 18\left[\frac{2}{3 \cdot 32}(16r^2+1)^{3/2}\right]_0^3$$
$$= \frac{3\pi}{4}(145^{3/2} - 1)$$

f. Evaluate the electric field $\vec{E} = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}, 0\right)$ on the paraboloid.

$$\vec{E}\left(\vec{R}(r,\theta)\right) = \left(\frac{r\cos\theta}{r^2}, \frac{r\sin\theta}{r^2}, 0\right) = \left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}, 0\right)$$

g. Compute the flux.

$$\iint \vec{E} \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 \vec{E} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^3 \left(-\frac{\cos\theta}{r}4r^2\cos\theta - \frac{\sin\theta}{r}4r^2\sin\theta\right) dr\, d\theta$$
$$= 2\pi\int_0^3 (-4r)\, dr = -8\pi\left[\frac{r^2}{2}\right]_0^3 = -36\pi$$

11. $(15$ points$)$ A cardboard box without a lid needs to hold 4000 cm^3. Find the dimensions of the box which uses the least cardboard.

Solution: Minimize the surface area $A = xy + 2xz + 2yz$ subject to the constraint that the volume is $V = xyz = 4000$.

Use Lagrange Multipliers

$\vec{\nabla}A = (y + 2z, x + 2z, 2x + 2y)$

$\vec{\nabla}V = (yz, xz, xy) \qquad \vec{\nabla}f = \lambda\vec{\nabla}g \qquad y + 2z = \lambda yz \qquad x + 2z = \lambda xz \qquad 2x + 2y = \lambda xy$

$$\lambda = \frac{y+2z}{yz} = \frac{x+2z}{xz} = \frac{2x+2y}{xy} \quad \Rightarrow \quad \frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

$$\Rightarrow \quad \frac{2}{y} = \frac{2}{x} \quad \text{and} \quad \frac{1}{z} = \frac{2}{y} \quad \Rightarrow \quad x = y \quad \text{and} \quad z = \frac{y}{2}$$

$$V = xyz = (y)(y)\left(\frac{y}{2}\right) = \frac{y^3}{2} = 4000 \qquad y^3 = 8000 \qquad y = 20 \qquad x = 20 \qquad z = 10$$

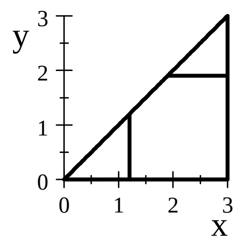
12. $(15$ points$)$ Draw the region of integration and compute $\int_0^3 \int_y^3 y\sqrt{x^3+9}\,dx\,dy$

Solution: To reverse the order of integration plot the region $0 \le y \le 3, \quad y \le x \le 3$.

Include a horizontal line to indicate the x limits.

Add a vertical line to indicate the new y limits.

Write the new integral and compute it.



$$\int_0^3 \int_0^x y\sqrt{x^3+9}\,dy\,dx = \int_0^3 \sqrt{x^3+9}\left[\frac{y^2}{2}\right]_0^x dx = \frac{1}{2}\int_0^3 \sqrt{x^3+9}\,x^2\,dx = \frac{1}{9}(x^3+9)^{3/2}\Big|_0^3$$

$$= \frac{1}{9}(36^{3/2} - 9^{3/2}) = \frac{216-27}{9} = 21$$