Name________________________ ID__________________

1-10	/50
11	/30
12	/ 5
13	/20
Total	/105

MATH 251 Final Exam Fall 2015

Sections 511 Version A Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. A triangle has vertices $A = (3,2,4)$, $B = (3,4,6)$ and $C = (6,-1,4)$. Find its area.

a. 3

b. $2\sqrt{3}$

c. $3\sqrt{3}$ Correct Choice

d. $6\sqrt{3}$

e. 6

Solution: $\overrightarrow{AB} = (0,2,2)$ $\overrightarrow{AC} = (3,-3,0)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & 2 \\ 3 & -3 & 0 \end{vmatrix} = 6\hat{\imath} + 6\jmath - 6k \qquad A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{6^2 + 6^2 + 6^2} = 3\sqrt{3}$$

2. Find the tangent plane to the graph of $z = x^3y + y^3x^2$ at $(x,y) = (1,2)$.
Where does it cross the z-axis?

a. -38 Correct Choice

b. -14

c. -10

d. 10

e. 38

Solution: $f(x,y) = x^3y + y^3x^2$ $f(1,2) = 10$

$f_x(x,y) = 3x^2y + 2y^3x$ $f_x(1,2) = 22$

$f_y(x,y) = x^3 + 3y^2x^2$ $f_y(1,2) = 13$

$$z = f_{\text{tan}}(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 10 + 22(x-1) + 13(y-2)$$

z-intercept $= f_{\text{tan}}(0,0) = 10 + 22(-1) + 13(-2) = -38$

3. Find the tangent plane to the graph of hyperbolic paraboloid

$$-4(x-3)^2 - 4(y-1)^2 + 9(z-2)^2 = 1$$

at the point $(4,2,1)$. Where does it cross the z-axis?

a. $-\frac{11}{9}$

b. $-\frac{11}{3}$

c. 0

d. $\frac{11}{3}$ Correct Choice

e. $\frac{11}{9}$

Solution: $P = (4,2,1)$ $F = -4(x-3)^2 - 4(y-1)^2 + 9(z-2)^2$

$\vec{\nabla} F = (-8(x-3), -8(y-1), 18(z-2))$ $\vec{N} = \vec{\nabla} F\big|_P = (-8,-8,-18)$

$\vec{N} \bullet X = \vec{N} \bullet P$ $-8x - 8y - 18z = -32 - 16 - 18 = -66$

z-intercept at $x = y = 0$: $-18z = -66$ $z = \frac{66}{18} = \frac{11}{3}$

4. Sketch the region of integration for the integral $\int_0^1 \int_{x^2}^1 x \sin(y^2)\, dy\, dx$ in problem (12), then select its value here:

a. $\frac{1}{4}\cos 1$

b. $\frac{1}{4}(1-\cos 1)$ Correct Choice

c. $\frac{1}{4}(\cos 1 - 1)$

d. $1 - \cos 1$

e. $\cos 1 - 1$

Solution: Reverse the order of integration:

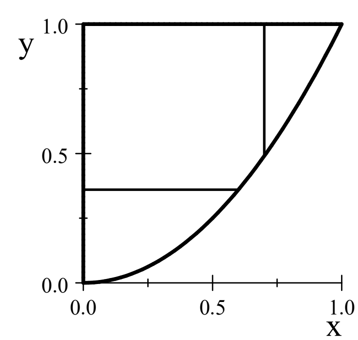
$$\int_0^1 \int_{x^2}^1 x \sin(y^2)\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} x \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} \sin(y^2) \right]_{x=0}^{\sqrt{y}} dy = \int_0^1 \frac{y}{2} \sin(y^2)\, dy$$

Substitute: $u = y^2$ $du = 2y\,dy$ $y\,dy = \frac{1}{2} du$

$$\int_0^1 \int_{x^2}^1 x \sin(y^2)\, dy\, dx = \frac{1}{4} \int \sin(u)\, du = \frac{-1}{4} \cos(u)$$

$$= \frac{-1}{4} \cos(y^2)\Big|_0^1 = \frac{1}{4}(1 - \cos 1)$$



5. Find the point on the curve $\vec{r}(t) = (50t - 3t^2, 25t - 4t^2)$ at which the speed is a minimum.

a. $(20, -15)$

b. $(80, -65)$

c. $(80, 65)$

d. $(175, 25)$ Correct Choice

e. $(325, 225)$

Solution: $\vec{v} = (50 - 6t, 25 - 8t)$ $\quad |\vec{v}| = \sqrt{(50-6t)^2 + (25-8t)^2} = 5\sqrt{4t^2 - 40t + 125}$

$$\frac{d|\vec{v}|}{dt} = \frac{5}{2}\frac{8t - 40}{\sqrt{4t^2 - 40t + 125}} = 0 \qquad t = 5 \qquad \vec{r}(5) = (250 - 75, 125 - 100) = (175, 25)$$

6. The surface of the Death star is a sphere of radius 2 with a hole cut out of one end, which we will take as centered at the south pole. It may be parametrized by

$$R(\phi,\theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$$

for $0 \le \phi \le \frac{2\pi}{3}$. Find the surface area.

a. $A = 12\pi$ Correct Choice

b. $A = 6\pi$

c. $A = 3\pi$

d. $A = 2\pi$

e. $A = \pi$

Solution: We first find the normal and its length:

$$\begin{matrix} & \\ \vec{e}_\phi = \\ \vec{e}_\theta = \end{matrix} \left| \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi) \\ (-2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0\ \ ) \end{matrix} \right|$$

$$\vec{N} = \vec{e}_\phi \times \vec{e}_\theta = \hat{\imath}(4\sin^2\phi\cos\theta) - \hat{\jmath}(-4\sin^2\phi\sin\theta) + \hat{k}(4\sin\phi\cos\phi\cos^2\theta - -4\sin\phi\cos\phi\sin^2\theta)$$
$$= \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$$

$$\left|\vec{N}\right| = \sqrt{(4\sin^2\phi\cos\theta)^2 + (4\sin^2\phi\sin\theta)^2 + (4\sin\phi\cos\phi)^2}$$
$$= \sqrt{16\sin^4\phi + 16\sin^2\phi\cos^2\phi} = 4\sin\phi$$

The surface area is

$$A = \iint 1\,dS = \int_0^{2\pi}\int_0^{2\pi/3} 4\sin\phi\,d\phi\,d\theta = 2\pi[-\cos\phi]_0^{2\pi/3}$$
$$= 8\pi\left(-\cos\frac{2\pi}{3} - -\cos 0\right) = 8\pi\left(- -\frac{1}{2} + 1\right) = 12\pi$$

7. Consider the solid below the cone given in cylindrical coordinates by $z = 4 - r$ above the xy-plane. Find the z-component of its centroid.

a. π

b. $\frac{1}{2}$

c. 1 Correct Choice

d. $\frac{32\pi}{3}$

e. $\frac{64\pi}{3}$

Solution: The volume is

$$V = \int_0^4 \int_0^{2\pi} \int_0^{4-r} 1\, r\, dz\, d\theta\, dr = 2\pi \int_0^4 [rz]_0^{4-r}\, dr = 2\pi \int_0^4 4r - r^2\, dr$$

$$= 2\pi\left[2r^2 - \frac{r^3}{3}\right]_0^4 = 2\pi\left(32 - \frac{64}{3}\right) = \frac{64\pi}{3}$$

The z-moment is

$$V_z = \int_0^4 \int_0^{2\pi} \int_0^{4-r} z\, r\, dz\, d\theta\, dr = 2\pi \int_0^4 \left[r\frac{z^2}{2}\right]_0^{4-r} dr = \pi \int_0^4 r(4-r)^2\, dr$$

$$= \pi \int_0^4 (16r - 8r^2 + r^3)\, dr = \pi\left[8r^2 - 8\frac{r^3}{3} + \frac{r^4}{4}\right]_0^4 = \pi\left(8 \cdot 4^2 - 8\frac{4^3}{3} + \frac{4^4}{4}\right)$$

$$= \pi 4^3\left(2 - 8\frac{1}{3} + 1\right) = \frac{64}{3}\pi$$

So the z-component of the centroid is

$$\bar{z} = \frac{V_z}{V} = 1$$

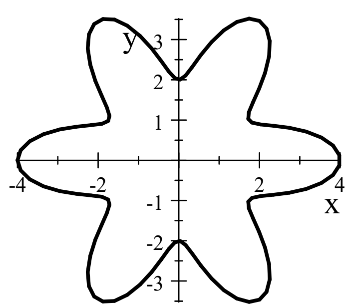
8. Compute $\oint_{\partial R} \vec{\nabla} f \cdot d\vec{s}$ for $f = ye^x$

counterclockwise around the polar curve

$$r = 3 + \cos(6\theta)$$

shown at the right.

Hint: Use a Theorem.



a. 0 Correct Choice

b. $6e^3$

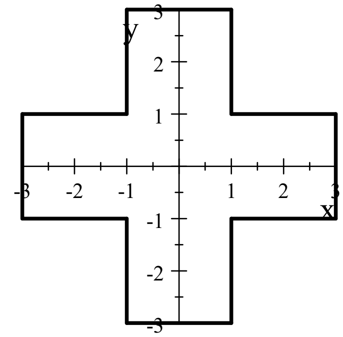
c. $3e^6$

d. $6e^6 - 3e^3$

e. $3e^3 - 6e^6$

Solution: By the FTCC, $\oint_{\partial R} \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = 0$ because $B = A$ no matter what point you start at.

9. Compute $\oint_{\partial P} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (5x^3 + 7y, 4x - 6y^4)$ counterclockwise around the complete boundary of the plus sign shown at the right. Hint: Use a Theorem.



a. 20

b. 3

c. -3

d. -30

e. -60 Correct Choice

Solution: $P = 5x^3 + 7y \qquad Q = 4x - 6y^4 \qquad \partial_x Q - \partial_y P = 4 - 7 = -3$

By Green's Theorem,

$$\oint_{\partial R} \vec{F} \cdot d\vec{s} = \iint_R (\partial_x Q - \partial_y P)\,dx\,dy = \iint_R -3\,dx\,dy = -3(\text{area}) = -3(20) = -60$$

10. Compute $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the cone $z = \sqrt{x^2 + y^2}$ for $z \le 4$ oriented down and out for $\vec{F} = (y\sqrt{z}, -x\sqrt{z}, \sqrt{z})$.

Hint: Use a Theorem.

a. 4

b. 8π

c. 16

d. 32

e. 64π Correct Choice

Solution: By Stokes' Theorem, $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$ oriented clockwise.

At the boundary, $z = \sqrt{x^2 + y^2} = 4$. So the boundary is the circle of radius $r = 4$ at height $z = 4$, parametrized by $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 4)$. The vector field is $\vec{F} = (8\sin\theta, -8\cos\theta, 2)$.

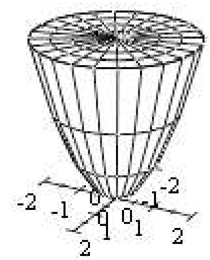
The tangent vector is $\vec{v} = (-4\sin\theta, 4\cos\theta, 0)$. Reversed $\vec{v} = (4\sin\theta, -4\cos\theta, 0)$.

So $\vec{F} \cdot \vec{v} = 32\sin^2\theta + 32\cos^2\theta = 32$. And the integral is

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\,dt = \int_0^{2\pi} 32\,dt = 64\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (30 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = \left(x^3 z, y^3 z, (x^2+y^2)^3\right)$ and the solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Be careful with orientations. Use the following steps:



Left Hand Side:

a. Compute the divergence:

$\vec{\nabla} \cdot \vec{F} = 3x^2 z + 3y^2 z = 3z(x^2+y^2)$

b. Express the divergence and the volume element in the appropriate coordinate system:

$\vec{\nabla} \cdot \vec{F} = 3zr^2 \qquad dV = r\,dr\,d\theta\,dz$

c. Compute the left hand side:

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{z}} 3zr^2\, r\,dr\,d\theta\,dz = 2\pi \int_0^4 \left[3z\frac{r^4}{4}\right]_{r=0}^{\sqrt{z}} dz$$

$$= \frac{3\pi}{2}\int_0^4 z^3\,dz = \frac{3\pi}{2}\frac{z^4}{4}\bigg|_0^4 = 96\pi$$

Right Hand Side:

The boundary surface consists of a paraboloid P and a disk D with appropriate orientations.

d. Parametrize the disk D:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4)$

e. Compute the tangent vectors:

$\vec{e}_r = (\cos\theta, \quad \sin\theta, \quad 0)$

$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$

f. Compute the normal vector:

$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (0,0,r)$

This is correctly up.

g. Evaluate $\vec{F} = \left(x^3 z, y^3 z, (x^2+y^2)^3\right)$ on the disk:

$\vec{F}\big|_{\vec{R}(r,\theta)} = (4r^3\cos^3\theta, 4r^3\sin^3\theta, r^6)$

h. Compute the dot product:

$\vec{F} \cdot \vec{N} = r^7$

i. Compute the flux through D:

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 r^7\,dr\,d\theta = 2\pi\left[\frac{r^8}{8}\right]_0^2 = 64\pi$$

The paraboloid P may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$

j. Compute the tangent vectors:

$$\vec{e}_r = \left(\cos\theta, \quad \sin\theta, \quad 2r\right)$$

$$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$$

k. Compute the normal vector:

$$\vec{N} = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(--2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$
$$= (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

This is oriented up. Need down. Reverse: $\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$

l. Evaluate $\vec{F} = \left(x^3z, y^3z, (x^2+y^2)^3\right)$ on the paraboloid:

$$\vec{F}\Big|_{\vec{R}(r,\theta)} = (r^5\cos^3\theta, r^5\sin^3\theta, r^6)$$

m. Compute the dot product:

$$\vec{F}\cdot\vec{N} = 2r^7\cos^4\theta + 2r^7\sin^4\theta - r^7 = r^7(2\cos^4\theta + 2\sin^4\theta - 1)$$

n. Compute the flux through P:

Hints: Use these integrals as needed, without proof:

$$\int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta = \pi \qquad \int_0^{2\pi}\sin^4\theta\,d\theta = \int_0^{2\pi}\cos^4\theta\,d\theta = \frac{3}{4}\pi \qquad \int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\pi$$

$$\iint_P \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^2 r^7(2\cos^4\theta + 2\sin^4\theta - 1)\,dr\,d\theta = \frac{r^8}{8}\Big|_0^2\left(2\int_0^{2\pi}\cos^4\theta\,d\theta + 2\int_0^{2\pi}\sin^4\theta\,d\theta - \int_0^{2\pi}1\,d\theta\right)$$
$$= 2^5\left(\frac{3}{2}\pi + \frac{3}{2}\pi - 2\pi\right) = 32\pi$$

o. Compute the **TOTAL** right hand side:

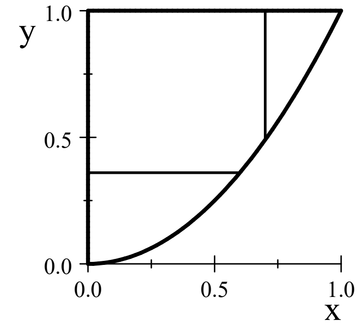
$$\iint_{\partial V}\vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot d\vec{S} + \iint_C \vec{F}\cdot d\vec{S} = 64\pi + 32\pi = 96\pi$$ which agrees with (c).

12. (5 points) Sketch the region of integration for the integral

$$\int_0^1 \int_{x^2}^1 x \sin(y^2)\, dy\, dx.$$

Shade in the region.

You computed its value in problem (4).



13. (20 points) Ham Duet is flying the Millenium Eagle through the galaxy. His current galactic position is $P = (1,4,4)$ lightyears. He is passing through a deadly polaron field whose density is $\delta = x^2 - yz$ polarons/lightyear^3.

a. If his current velocity is $\vec{v} = \langle 0.3, 0.2, 0.1 \rangle$ lightyears/year, at what rate does he see the polaron density changing?

Solution: $\vec{\nabla}\delta = (2x, -z, -y)$ $\quad \vec{\nabla}\delta\Big|_P = (2,-4,-4)$

$$\frac{d\delta}{dt} = \vec{v} \cdot \vec{\nabla}\delta\Big|_P = 0.3 \cdot 2 - 0.2 \cdot 4 - 0.1 \cdot 4 = -0.6$$ polarons/lightyear^3/year.

b. Ham decides to change his velocity to get out of the polaron field. If the Millenium Eagle's maximum speed is 0.9 lightyears/year, with what velocity should Ham travel to reduce the polaron's density as fast as possible?

Solution: The direction of maximum increase is $\vec{\nabla}\delta\Big|_P = (2,-4,-4)$.

The direction of maximum decrease is $-\vec{\nabla}\delta\Big|_P = (-2,4,4)$.

The length is $\left|\vec{\nabla}\delta\Big|_P\right| = \sqrt{4+16+16} = 6$.

So the unit vector of maximum decrease is $\hat{v} = \dfrac{-\vec{\nabla}\delta\Big|_P}{\left|\vec{\nabla}\delta\Big|_P\right|} = \dfrac{1}{6}(-2,4,4) = \left(-\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$.

Since the maximum speed is $|\vec{v}| = 0.9$, Ham's velocity should be

$$\vec{v} = |\vec{v}|\hat{v} = 0.9\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = (-0.3, 0.6, 0.6).$$