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MATH 251 Final Exam Fall 2015

Sections 511 Version B Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1-10	/50
11	/30
12	/ 5
13	/20
Total	/105

1. A triangle has vertices $A = (3,2,4)$, $B = (3,4,6)$ and $C = (6,-1,4)$. Find the angle at A.

a. $30°$

b. $45°$

c. $60°$

d. $120°$ Correct Choice

e. $150°$

Solution: $\overrightarrow{AB} = (0,2,2)$ $\overrightarrow{AC} = (3,-3,0)$

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{-6}{2\sqrt{2}\,3\sqrt{2}} = \frac{-1}{2} \qquad \theta = 120°$$

2. Find the tangent plane to the graph of $z = x^2y + y^3x^3$ at $(x,y) = (2,1)$.
Where does it cross the z-axis?

a. -72

b. -48 Correct Choice

c. -12

d. 12

e. 48

Solution:
$$f(x,y) = x^2y + y^3x^3 \qquad f(2,1) = 12$$
$$f_x(x,y) = 2xy + 3y^3x^2 \qquad f_x(2,1) = 16$$
$$f_y(x,y) = x^2 + 3y^2x^3 \qquad f_y(2,1) = 28$$

$$z = f_{\tan}(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 12 + 16(x-2) + 28(y-1)$$

z-intercept $= f_{\tan}(0,0) = 12 + 16(-2) + 28(-1) = -48$

3. Find the tangent plane to the graph of hyperbolic paraboloid

$$9(x-3)^2 - 4(y-1)^2 - 4(z-2)^2 = 1$$

at the point $(2,2,3)$. Where does it cross the z-axis?

a. $\frac{19}{2}$ Correct Choice

b. 19

c. 0

d. -19

e. $-\frac{19}{2}$

Solution: $P = (2,2,3)$ $F = 9(x-3)^2 - 4(y-1)^2 - 4(z-2)^2$

$\vec{\nabla}F = (18(x-3), -8(y-1), -8(z-2))$ $\vec{N} = \vec{\nabla}F\Big|_P = (-18, -8, -8)$

$\vec{N} \cdot X = \vec{N} \cdot P$ $-18x - 8y - 8z = -18 \cdot 2 - 8 \cdot 2 - 8 \cdot 3 = -76$

z-intercept at $x = y = 0$: $-8z = -76$ $z = \frac{76}{8} = \frac{19}{2}$

4. Sketch the region of integration for the integral $\int_0^{2\sqrt{2}} \int_x^{\sqrt{16-x^2}} e^{x^2+y^2}\, dy\, dx$ in problem (12), then select its value here:

a. $\frac{\pi}{4}(e^{16} - 1)$

b. $\frac{\pi}{4}(e^{8} - 1)$

c. $\frac{\pi}{8}(e^{16} - 1)$ Correct Choice

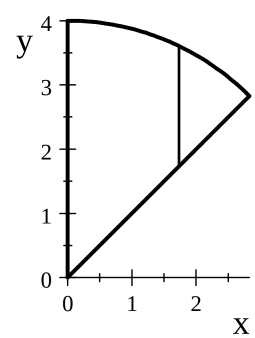
d. $\frac{\pi}{8}(e^{8} - 1)$

e. $\frac{\pi}{16}(e^{16} - 1)$

Solution: Switch to polar coordinates:

$$\int_{\pi/4}^{\pi/2} \int_0^4 e^{r^2} r\, dr\, d\theta = \left[\theta\right]_{\pi/4}^{\pi/2} \left[\frac{1}{2} e^{r^2}\right]_0^4$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{4}\right) \frac{1}{2} (e^{16} - 1) = \frac{\pi}{8}(e^{16} - 1)$$



5. Find the line perpendicular to the curve $x^3 + y^3 - xy = 7$ at the point $(2,1)$.

a. $(11 - 2t, 1 - t)$

b. $(2 + t, 1 + 11t)$

c. $(2 + t, 1 - 11t)$

d. $(2 + 11t, 1 - t)$

e. $(2 + 11t, 1 + t)$ Correct Choice

Solution: A vector perpendicular to the curve is the gradient at the point:

$\vec{\nabla} f = (3x^2 - y, 3y^2 - x) \qquad \vec{v} = \vec{\nabla} f\Big|_{(2,1)} = (12 - 1, 3 - 2) = (11,1)$

So the perpendicular line is $X = P + t\vec{v}$ or $(x,y) = (2,1) + t(11,1) = (2 + 11t, 1 + t)$

6. Consider the surface of the cone given in cylindrical coordinates by $z = 4 - r$ above the xy-plane. It may be parametrized by

$$R(r,\theta) = (r\cos\theta, r\sin\theta, 4 - r).$$

Find the z-component of its centroid.

a. $\dfrac{1}{3}$

b. $\dfrac{2}{3}$

c. $\dfrac{4}{3}$ Correct Choice

d. $\dfrac{32\sqrt{2}\,\pi}{3}$

e. $16\sqrt{2}\,\pi$

Solution: We first find the normal and its length:

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\cos\theta & \sin\theta & -1) \\ (-r\sin\theta & r\cos\theta & 0\ \ ) \end{array} \right|$$

$$\vec{N} = \vec{e}_\phi \times \vec{e}_\theta = \hat{\imath}(- -r\cos\theta) - \hat{\jmath}(-r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$$

$$\left|\vec{N}\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2}\,r$$

The surface area is

$$A = \int_0^{2\pi}\int_0^4 \sqrt{2}\,r\,dr\,d\theta = 2\pi\sqrt{2}\left[\frac{r^2}{2}\right]_0^4 = 16\sqrt{2}\,\pi$$

Since $z = 4 - r$, the z-moment is

$$A_z = \int_0^{2\pi}\int_0^4 (4 - r)\sqrt{2}\,r\,dr\,d\theta = 2\pi\sqrt{2}\int_0^4 (4r - r^2)\,dr$$

$$= 2\pi\sqrt{2}\left[2r^2 - \frac{r^3}{3}\right]_0^4 = 2\pi\sqrt{2}\left(32 - \frac{64}{3}\right) = \frac{64\sqrt{2}\,\pi}{3}$$

So the z-component of the centroid is

$$\bar{z} = \frac{A_z}{A} = \frac{64\sqrt{2}\,\pi}{3}\,\frac{1}{16\sqrt{2}\,\pi} = \frac{4}{3}$$

7. Death star is basically a spherical shell with a hole cut out of one end, which we will take as centered at the south pole. In spherical coordinates, it fills the region between $1 \le \rho \le 4$ and $0 \le \phi \le \frac{2\pi}{3}$. Find the volume.

a. $V = 21\pi$

b. $V = 63\pi$ Correct Choice

c. $V = 84\pi$

d. $V = 105\pi$

e. $V = 126\pi$

Solution: The volume is

$$V = \int_0^{2\pi}\int_0^{2\pi/3}\int_1^4 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi[-\cos\phi]_0^{2\pi/3}\left[\frac{\rho^3}{3}\right]_1^4$$
$$= 2\pi\left[-\cos\frac{2\pi}{3} - -\cos 0\right]\left[\frac{4^3-1}{3}\right] = 2\pi\left(- -\frac{1}{2} - -1\right)21 = 63\pi$$

8. Compute $\oint_{\partial R} \vec{\nabla} f \cdot d\vec{s}$ for $f = ye^x$

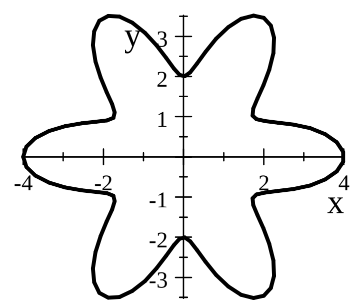
counterclockwise around

the polar curve

$r = 3 + \cos(6\theta)$

shown at the right.

Hint: Use a Theorem.



a. $6e^3$

b. $3e^6$

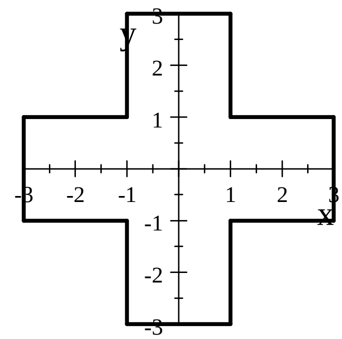
c. $6e^6 - 3e^3$

d. $3e^3 - 6e^6$

e. 0 Correct Choice

Solution: By the FTCC, $\oint_{\partial R} \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = 0$ because $B = A$ no matter what point you start at.

9. Compute $\oint_{\partial P} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (3x^3 + 5y, 7x - 4y^4)$ counterclockwise around the complete boundary of the plus sign shown at the right.
Hint: Use a Theorem.



a. -40

b. -20

c. 20

d. 40 Correct Choice

e. 240

Solution: $P = 3x^3 + 5y$ $Q = 7x - 4y^4$ $\partial_x Q - \partial_y P = 7 - 5 = 2$
By Green's Theorem,

$$\oint_{\partial R} \vec{F} \cdot d\vec{s} = \iint_R (\partial_x Q - \partial_y P)\, dx\, dy = \iint_R 2\, dx\, dy = 2\big(\text{area}\big) = 2(20) = 40$$

10. Compute $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the paraboloid $z = 4(x^2 + y^2)$ for $z \le 16$ oriented down and out for $\vec{F} = (y\sqrt{z}, -x\sqrt{z}, \sqrt{z})$.
Hint: Use a Theorem.

a. 64

b. 32π Correct Choice

c. 16

d. 8π

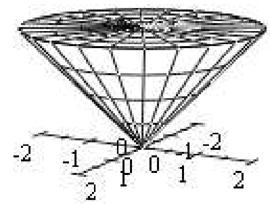
e. 4

Solution: By Stokes' Theorem, $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial P} \vec{F} \cdot d\vec{s}$ oriented clockwise.
At the boundary, $z = 4(x^2 + y^2) = 16$. So the boundary is the circle of radius $r = 2$ at height $z = 16$, parametrized by $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 16)$. The vector field is $\vec{F} = (8\sin\theta, -8\cos\theta, 4)$.
The tangent vector is $\vec{v} = (-2\sin\theta, 2\cos\theta, 0)$. Reversed $\vec{v} = (2\sin\theta, -2\cos\theta, 0)$.
So $\vec{F} \cdot \vec{v} = 16\sin^2\theta + 16\cos^2\theta = 16$. And the integral is

$$\oint_{\partial P} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, dt = \int_0^{2\pi} 16\, dt = 32\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (30 points) Verify Gauss' Theorem $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$



for the vector field $\vec{F} = \left(x^3 z, y^3 z, (x^2+y^2)^2\right)$ and the solid between the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$.

Be careful with orientations. Use the following steps:

Left Hand Side:

a. Compute the divergence:

$$\vec{\nabla} \cdot \vec{F} = 3x^2 z + 3y^2 z = 3z(x^2+y^2)$$

b. Express the divergence and the volume element in the appropriate coordinate system:

$$\vec{\nabla} \cdot \vec{F} = 3zr^2 \qquad\qquad dV = r\,dr\,d\theta\,dz$$

c. Compute the left hand side:

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^2 \int_0^{2\pi} \int_0^z 3zr^2\, r\,dr\,d\theta\,dz = 2\pi \int_0^2 \left[3z\frac{r^4}{4}\right]_{r=0}^{z} dz$$

$$= \frac{3\pi}{2}\int_0^2 z^5\,dz = \frac{3\pi}{2}\frac{z^6}{6}\bigg|_0^2 = \frac{\pi}{4}2^6 = 2^4\pi = 16\pi$$

Right Hand Side:

The boundary surface consists of a cone C and a disk D with appropriate orientations.

d. Parametrize the disk D:

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2)$$

e. Compute the tangent vectors:

$$\vec{e}_r = (\cos\theta, \quad \sin\theta, \quad 0)$$

$$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$$

f. Compute the normal vector:

$$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (0,0,r)$$

This is correctly up.

g. Evaluate $\vec{F} = \left(x^3 z, y^3 z, (x^2+y^2)^2\right)$ on the disk:

$$\vec{F}\Big|_{\vec{R}(r,\theta)} = (2r^3\cos^3\theta, 2r^3\sin^3\theta, r^4)$$

h. Compute the dot product:

$$\vec{F} \cdot \vec{N} = r^5$$

i. Compute the flux through D:

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 r^5\,dr\,d\theta = 2\pi\left[\frac{r^6}{6}\right]_0^2 = \frac{64}{3}\pi$$

The cone C may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

j. Compute the tangent vectors:

$$\vec{e}_r = (\cos\theta, \quad \sin\theta, \quad 1)$$

$$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$$

k. Compute the normal vector:

$$\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(- -r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$

$$= (-r\cos\theta, -r\sin\theta, r)$$

This is oriented up. Need down. Reverse: $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

l. Evaluate $\vec{F} = \left(x^3 z, y^3 z, (x^2+y^2)^2\right)$ on the cone:

$$\vec{F}\Big|_{\vec{R}(r,\theta)} = (r^4\cos^3\theta, r^4\sin^3\theta, r^4)$$

m. Compute the dot product:

$$\vec{F} \bullet \vec{N} = r^5\cos^4\theta + r^5\sin^4\theta - r^5 = r^5(\cos^4\theta + \sin^4\theta - 1)$$

n. Compute the flux through C:

Hints: Use these integrals as needed, without proof:

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \int_0^{2\pi} \cos^2\theta\, d\theta = \pi \qquad \int_0^{2\pi} \sin^4\theta\, d\theta = \int_0^{2\pi} \cos^4\theta\, d\theta = \frac{3}{4}\pi \qquad \int_0^{2\pi} \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{4}\pi$$

$$\iint_C \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^2 r^5(\cos^4\theta + \sin^4\theta - 1)\, dr\, d\theta = \frac{r^6}{6}\Big|_0^2 \left(\int_0^{2\pi} \cos^4\theta\, d\theta + \int_0^{2\pi} \sin^4\theta\, d\theta - \int_0^{2\pi} 1\, d\theta\right)$$

$$= \frac{2^5}{3}\left(\frac{3}{4}\pi + \frac{3}{4}\pi - 2\pi\right) = -\frac{16}{3}\pi$$

o. Compute the **TOTAL** right hand side:

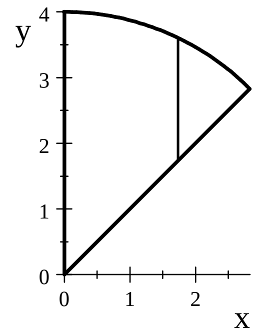
$$\iint_{\partial V} \vec{F} \bullet d\vec{S} = \iint_D \vec{F} \bullet d\vec{S} + \iint_C \vec{F} \bullet d\vec{S} = \frac{64}{3}\pi - \frac{16}{3}\pi = 16\pi$$ which agrees with (c).

12. (5 points) Sketch the region of integration for the integral

$$\int_0^{2\sqrt{2}} \int_x^{\sqrt{16-x^2}} e^{x^2+y^2}\, dy\, dx.$$

Shade in the region.

You computed its value in problem (4).



13. (20 points) A cardboard box needs to hold 96 cm^3. The cardboard for the vertical sides costs 12¢ per cm^2 while the thicker bottom costs 36¢ per cm^2. There is no top. What are the length, width, height and cost of the box which costs the least?

Solution: The volume constraint is $V = LWH = 96$. The cost is $C = 36LW + 12(2LH + 2WH) = 36LW + 24LH + 24WH$ which needs to be minimized.

Lagrange Multipliers:

$\vec{\nabla} C = (36W + 24H, 36L + 24H, 24L + 24W)$ $\quad \vec{\nabla} V = (WH, LH, LW)$ $\quad \vec{\nabla} C = \lambda \vec{\nabla} V$

$36W + 24H = \lambda WH \qquad 36LW + 24LH = \lambda LWH$

$36L + 24H = \lambda LH \qquad 36LW + 24WH = \lambda LWH \qquad 36LW + 24LH = 36LW + 24WH$

$24LH = 24WH \qquad L = W$

$24L + 24W = \lambda LW \qquad 24LH + 24WH = \lambda LWH \qquad 36LW + 24WH = 24LH + 24WH$

$36LW = 24LH \qquad 3W = 2H$

$LWH = WW\frac{3}{2}W = 96 \qquad W^3 = 64 \qquad W = 4 \qquad L = 4 \qquad H = 6$

$C = 36LW + 24LH + 24WH = 36 \cdot 4 \cdot 4 + 24 \cdot 4 \cdot 6 + 24 \cdot 4 \cdot 6 = 1728$¢