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1-10	/50
11	/30
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Total	/105

MATH 251 Final Exam Fall 2015

Sections 512 Version B Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. A triangle has vertices $A = (2,2,1)$, $B = (3,4,2)$ and $C = (2,5,4)$. Find its area.

a. $\frac{3}{2}$

b. $\frac{3}{2}\sqrt{3}$ Correct Choice

c. $\sqrt{3}$

d. $3\sqrt{3}$

e. 3

Solution: $\overrightarrow{AB} = (1,2,1)$ $\overrightarrow{AC} = (0,3,3)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 3 & 3 \end{vmatrix} = 3\hat{\imath} - 3\jmath + 3k \qquad A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{3^2 + 3^2 + 3^2} = \frac{3}{2}\sqrt{3}$$

2. Find the tangent plane to the graph of $z = x^3y + y^3x^2$ at $(x,y) = (1,2)$.
Where does it cross the z-axis?

a. -38 Correct Choice

b. -14

c. -10

d. 10

e. 38

Solution: $f(x,y) = x^3y + y^3x^2 \quad f(1,2) = 10$

$f_x(x,y) = 3x^2y + 2y^3x \quad f_x(1,2) = 22$

$f_y(x,y) = x^3 + 3y^2x^2 \quad f_y(1,2) = 13$

$$z = f_{\tan}(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 10 + 22(x-1) + 13(y-2)$$

z-intercept $= f_{\tan}(0,0) = 10 + 22(-1) + 13(-2) = -38$

3. Find the tangent plane to the graph of hyperbolic paraboloid

$$-4(x-3)^2 - 4(y-1)^2 + 9(z-2)^2 = 1$$

at the point $(4,2,1)$. Where does it cross the z-axis?

a. $-\dfrac{11}{9}$

b. $-\dfrac{11}{3}$

c. 0

d. $\dfrac{11}{3}$ Correct Choice

e. $\dfrac{11}{9}$

Solution: $P = (4,2,1)$ $F = -4(x-3)^2 - 4(y-1)^2 + 9(z-2)^2$

$\vec{\nabla}F = (-8(x-3), -8(y-1), 18(z-2))$ $\vec{N} = \vec{\nabla}F\Big|_P = (-8,-8,-18)$

$\vec{N} \cdot X = \vec{N} \cdot P$ $-8x - 8y - 18z = -32 - 16 - 18 = -66$

z-intercept at $x = y = 0$: $-18z = -66$ $z = \dfrac{66}{18} = \dfrac{11}{3}$

4. Queen Lean is flying the Millenium Eagle through the galaxy. Her current galactic position is $P = (4,3,1)$ lightyears and her current velocity is $\vec{v} = \langle 0.1, 0.2, 0.3 \rangle$ lightyears/year. She is passing through a deadly polaron field whose density δ is related to the dark energy intensity I and the dark matter pressure P by $\delta = IP$.

She measures the dark energy intensity and its gradient are currently

$$I = 7 \text{ lumens and } \vec{\nabla}I = \langle 2,3,1 \rangle \text{ lumens /lightyear}$$

She measures the dark matter pressure and its gradient are currently

$$P = 0.8 \text{ dynes/lightyear}^2 \text{ and } \vec{\nabla}P = \langle 0.4, 0.1, 0.2 \rangle \text{ dynes/lightyear}^3$$

At what rate does she see the polaron density changing?

a. $\dfrac{d\delta}{dt} = -7.724$

b. $\dfrac{d\delta}{dt} = -1.22$

c. $\dfrac{d\delta}{dt} = -1.06$

d. $\dfrac{d\delta}{dt} = 1.06$ Correct Choice

e. $\dfrac{d\delta}{dt} = 7.724$

Solution:

$$\frac{dI}{dt} = \vec{\nabla}I \cdot \vec{v} = \langle 2,3,1 \rangle \cdot \langle 0.1, 0.2, 0.3 \rangle = .2 + .6 + .3 = 1.1$$

$$\frac{dP}{dt} = \vec{\nabla}P \cdot \vec{v} = \langle 0.4, 0.1, 0.2 \rangle \cdot \langle 0.1, 0.2, 0.3 \rangle = .04 + .02 + .06 = .12$$

$$\frac{d\delta}{dt} = \frac{\partial\delta}{\partial I}\frac{dI}{dt} + \frac{\partial\delta}{\partial P}\frac{dP}{dt} = P\frac{dI}{dt} + I\frac{dP}{dt} = 0.2 \cdot 1.1 + 7 \cdot .12 = 1.06$$

5. Sketch the region of integration for the integral $\int_0^2 \int_y^{\sqrt{8-y^2}} e^{x^2+y^2}\,dx\,dy$ in problem (12),

then select its value here:

a. $\frac{\pi}{4}(e^8 - 1)$

b. $\frac{\pi}{8}(e^{16} - 1)$

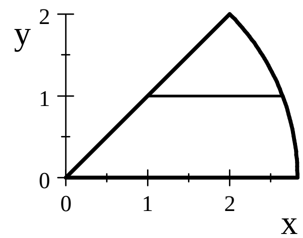
c. $\frac{\pi}{8}(e^8 - 1)$ Correct Choice

d. $\frac{\pi}{16}(e^{16} - 1)$

e. $\frac{\pi}{16}(e^8 - 1)$

Solution: Switch to polar coordinates:

$$\int_0^{\pi/4} \int_0^{2\sqrt{2}} e^{r^2} r\,dr\,d\theta = \Big[\theta\Big]_0^{\pi/4} \Big[\frac{1}{2} e^{r^2}\Big]_0^{2\sqrt{2}}$$
$$= \left(\frac{\pi}{4}\right)\frac{1}{2}(e^8 - 1) = \frac{\pi}{8}(e^8 - 1)$$



6. The surface of the Death star is a sphere of radius 2 with a hole cut out of one end, which we will take as centered at the south pole. It may be parametrized by

$$R(\phi,\theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$$

for $0 \le \phi \le \frac{2\pi}{3}$. Find the surface area.

a. $A = \pi$

b. $A = 2\pi$

c. $A = 3\pi$

d. $A = 6\pi$

e. $A = 12\pi$ Correct Choice

Solution: We first find the normal and its length:

$$\begin{array}{r} \\ \vec{e}_\phi = \\ \vec{e}_\theta = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi) \\ (-2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0\ \ ) \end{array} \right|$$

$$\vec{N} = \vec{e}_\phi \times \vec{e}_\theta = \hat{\imath}(4\sin^2\phi\cos\theta) - \hat{\jmath}(-4\sin^2\phi\sin\theta) + \hat{k}(4\sin\phi\cos\phi\cos^2\theta - -4\sin\phi\cos\phi\sin^2\theta)$$
$$= \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$$

$$\left|\vec{N}\right| = \sqrt{(4\sin^2\phi\cos\theta)^2 + (4\sin^2\phi\sin\theta)^2 + (4\sin\phi\cos\phi)^2}$$
$$= \sqrt{16\sin^4\phi + 16\sin^2\phi\cos^2\phi} = 4\sin\phi$$

The surface area is

$$A = \iint 1\,dS = \int_0^{2\pi}\int_0^{2\pi/3} 4\sin\phi\,d\phi\,d\theta = 2\pi[-\cos\phi]_0^{2\pi/3}$$
$$= 8\pi\left(-\cos\frac{2\pi}{3} - -\cos 0\right) = 8\pi\left(- -\frac{1}{2} + 1\right) = 12\pi$$

7. Consider the solid below the cone given in cylindrical coordinates by $z = 4 - r$ above the xy-plane. Its temperature is $T = z$. Find its average temperature.

a. π

b. 1 Correct Choice

c. $\frac{1}{2}$

d. $\frac{32\pi}{3}$

e. $\frac{64\pi}{3}$

Solution: The volume is

$$V = \int_0^4 \int_0^{2\pi} \int_0^{4-r} 1\, r\, dz\, d\theta\, dr = 2\pi \int_0^4 [rz]_0^{4-r}\, dr = 2\pi \int_0^4 4r - r^2\, dr$$
$$= 2\pi\left[2r^2 - \frac{r^3}{3}\right]_0^4 = 2\pi\left(32 - \frac{64}{3}\right) = \frac{64\pi}{3}$$

The integral of the temperature, $T = z$, is

$$\iiint T\, dV = \int_0^4 \int_0^{2\pi} \int_0^{4-r} z\, r\, dz\, d\theta\, dr = 2\pi \int_0^4 \left[r\frac{z^2}{2}\right]_0^{4-r} dr = \pi \int_0^4 r(4-r)^2\, dr$$
$$= \pi \int_0^4 (16r - 8r^2 + r^3)\, dr = \pi\left[8r^2 - 8\frac{r^3}{3} + \frac{r^4}{4}\right]_0^4 = \pi\left(8 \cdot 4^2 - 8\frac{4^3}{3} + \frac{4^4}{4}\right)$$
$$= \pi 4^3\left(2 - 8\frac{1}{3} + 1\right) = \frac{64}{3}\pi$$

So the average temperature is

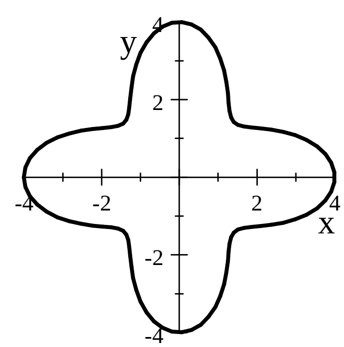
$$T_{\text{ave}} = \frac{1}{V} \iiint T\, dV = 1$$

8. Compute $\oint_{\partial R} \vec{\nabla} f \cdot d\vec{s}$ for $f = xe^y$ counterclockwise around the polar curve

$r = 3 + \cos(4\theta)$

shown at the right.

Hint: Use a Theorem.



a. 0 Correct Choice

b. $3e^4$

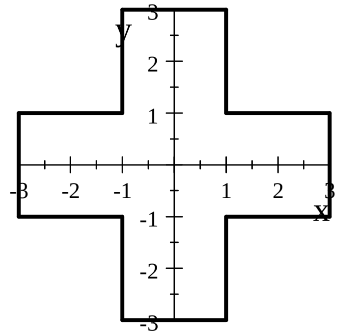
c. $4e^3$

d. $3e^3 - 4e^4$

e. $4e^4 - 3e^3$

Solution: By the FTCC, $\oint_{\partial R} \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = 0$ because $B = A$ no matter what point you start at.

9. Compute $\oint_{\partial P} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (8x^3 + 5y, x - 5y^4)$ counterclockwise around the complete boundary of the plus sign shown at the right.
Hint: Use a Theorem.



a. -20

b. -40

c. -80 Correct Choice

d. 20

e. 80

Solution: $P = 8x^3 + 5y \qquad Q = x - 5y^4 \qquad \partial_x Q - \partial_y P = 1 - 5 = -4$
By Green's Theorem,

$$\oint_{\partial R} \vec{F} \cdot d\vec{s} = \iint_R (\partial_x Q - \partial_y P)\, dx\, dy = \iint_R -4\, dx\, dy = -4(\text{area}) = -4(20) = -80$$

10. Compute $\iint_{\partial V} \vec{F} \cdot d\vec{S}$ over the complete surface of the hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$ oriented outward,for $\vec{F} = \left(x^3 z, y^3 z, \frac{3}{4} z^4\right)$
Hint: Use a Theorem.

a. -128π

b. -64π

c. -32π

d. 32π Correct Choice

e. 64π

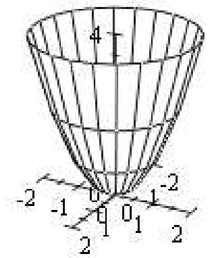
Solution: By Gauss' Theorem $\iint_{\partial H} \vec{F} \cdot d\vec{S} = \iiint_H \vec{\nabla} \cdot \vec{F}\, dV$. The divergence is $\vec{\nabla} \cdot \vec{F} = 3x^2 z + 3y^2 z + 3z^3 = 3z(x^2 + y^2 + z^2) = 3\rho^3 \cos\phi$ and the volume element is $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.
So the integral is:

$$\iiint_H \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 3\rho^3 \cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \left[\frac{\sin^2\phi}{2} \right]_0^{\pi/2} \left[3\frac{\rho^6}{6} \right]_0^2 = 2^5\pi = 32\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (30 points) Verify Stokes' Theorem $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial P} \vec{F} \cdot d\vec{s}$



for the vector field $\vec{F} = (yz^2, -xz^2, z^3)$ and the surface of the paraboloid $z = x^2 + y^2$ for $z \le 4$, oriented down and out. Be careful with orientations. Use the following steps:

Left Hand Side:

The paraboloid, P, may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$

a. Compute the tangent vectors:

$\vec{e}_r = (\cos\theta, \quad \sin\theta, \quad 2r)$

$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$

b. Compute the normal vector:

$\vec{N} = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(- -2r^2\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$

This is oriented up and in. Need down and out. Reverse: $\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$

c. Compute the curl of the vector field $\vec{F} = (yz^2, -xz^2, z^3)$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz^2 & -xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0 - -2xz) - \hat{\jmath}(0 - 2yz) + \hat{k}(-z^2 - z^2) = (2xz, 2yz, -2z^2)$$

d. Evaluate the curl of $\vec{F}$ on the paraboloid:

$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = (2r^3\cos\theta, 2r^3\sin\theta, -2r^4)$

e. Compute the dot product of the curl of $\vec{F}$ and the normal $\vec{N}$.

$\vec{\nabla} \times \vec{F} \cdot \vec{N} = 4r^5\cos^2\theta + 4r^5\sin^2\theta + 2r^5 = 6r^5$

f. Compute the left hand side:

$$\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_P \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (6r^5)\, dr\, d\theta = 2\pi\left[r^6\right]_0^2 = 128\pi$$

Right Hand Side:

g. Parametrize the circle, ∂C:

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4)$

h. Find the tangent vector on the curve:

$\vec{v} = (-2\sin\theta, 2\cos\theta, 0)$

This is counterclockwise. Need clockwise. Reverse $\vec{v} = (2\sin\theta, -2\cos\theta, 0)$

i. Evaluate $\vec{F} = (yz^2, -xz^2, z^3)$ on the circle:

$\vec{F}\Big|_{\vec{r}(\theta)} = (32\sin\theta, -32\cos\theta, 64)$

j. Compute the dot product of $\vec{F}$ and the tangent vector $\vec{v}$:

$\vec{F} \cdot \vec{v} = 64\sin^2\theta + 64\cos^2\theta = 64$

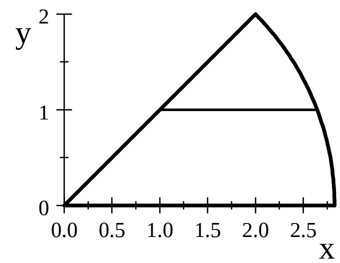
k. Compute the right hand side:

$\int_{\partial P} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 64\, d\theta = 128\pi$ which agrees with (f).

12. (5 points) Sketch the region of integration for the integral

$$\int_0^2 \int_y^{\sqrt{8-y^2}} e^{x^2+y^2}\, dx\, dy.$$

You computed its value in problem (5).



13. Duke Skywater is flying the Millenium Eagle through the galaxy. His current galactic position is $P = (4,4,1)$ lightyears. He is passing through a deadly polaron field whose density is $\delta = xy - z^2$ polarons/lightyear^3.

a. If his current velocity is $\vec{v} = \langle 0.3, 0.2, 0.1 \rangle$ lightyears/year, at what rate does he see the polaron density changing?

Solution: $\vec{\nabla}\delta = (y, x, -2z)$ $\quad \vec{\nabla}\delta\big|_P = (4,4,-2)$

$$\frac{d\delta}{dt} = \vec{v} \cdot \vec{\nabla}\delta\big|_P = 0.3 \cdot 4 + 0.2 \cdot 4 - 0.1 \cdot 2 = 1.8$$ polarons/lightyear^3/year.

b. Duke decides to change his velocity to get out of the polaron field. If the Millenium Eagle's maximum speed is 0.9 lightyears/year, with what velocity should Duke travel to reduce the polaron's density as fast as possible?

Solution: The direction of maximum increase is $\vec{\nabla}\delta\big|_P = (4,4,-2)$.

The direction of maximum decrease is $-\vec{\nabla}\delta\big|_P = (-4,-4,2)$.

The length is $\left|\vec{\nabla}\delta\big|_P\right| = \sqrt{16+16+4} = 6$.

So the unit vector of maximum decrease is $\hat{v} = \dfrac{-\vec{\nabla}\delta\big|_P}{\left|\vec{\nabla}\delta\big|_P\right|} = \frac{1}{6}(-4,-4,2) = \left(-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$.

Since the maximum speed is $|\vec{v}| = 0.9$, Duke's velocity should be

$\vec{v} = |\vec{v}|\hat{v} = 0.9\left(-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) = (-0.6, -0.6, 0.3)$.