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MATH 251 Exam 1 Version B Fall 2018

Sections 504/505 Solutions P. Yasskin

1-9	/54	11	/16
10	/36	Total	/106

Multiple Choice: (6 points each. No part credit.)

1. The Galactic Federation is trying to keep a stasis pod stationary in intergalactic space where there is no gravity. They already have 2 tractor beams pulling on the pod with the forces

$$\vec{F}_1 = \langle 4,1,-3\rangle \quad \text{and} \quad \vec{F}_2 = \langle -2,2,1\rangle$$

If they apply a 3^{rd} tractor beam on the pod, what should its force $\vec{F}_3$ be to keep the pod stationary?

a. $\vec{F}_3 = \langle 2,3,2\rangle$

b. $\vec{F}_3 = \langle 2,-3,-2\rangle$

c. $\vec{F}_3 = \langle 2,3,-2\rangle$

d. $\vec{F}_3 = \langle -2,3,2\rangle$

e. $\vec{F}_3 = \langle -2,-3,2\rangle$ Correct Choice

Solution: To balance forces, $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$. So

$$\vec{F}_3 = -\vec{F}_1 - \vec{F}_2 = -\langle 4,1,-3\rangle - \langle -2,2,1\rangle = \langle -2,-3,2\rangle$$

2. The Galactic Federation moves a stasis pod from $(2,3,4)$ to $(6,9,0)$ by applying the 2 forces:

$$\vec{F}_1 = \langle 4,1,-3\rangle \quad \text{and} \quad \vec{F}_2 = \langle -2,2,1\rangle$$

How much work is done by the force $\vec{F}_1$ only?

a. $W = 10$

b. $W = 22$

c. $W = 33$

d. $W = 34$ Correct Choice

e. $W = 0$

Solution: The displacement is $\vec{D} = (6,9,0) - (2,3,4) = \langle 4,6,-4\rangle$. So the work is

$$W = \vec{F}_1 \cdot \vec{D} = 16 + 6 + 12 = 34$$

3. If $\vec{u}$ points Up and $\vec{v}$ points SouthEast, where does $\vec{u} \times \vec{v}$ point?

a. Down

b. NorthWest

c. NorthEast Correct Choice

d. SouthWest

e. South

Solution: Hold your right hand with the fingers pointing Up and the palm facing SouthEast. Then the thumb points NorthEast.

4. Convert the polar equation $r = 4\cos\theta$ to rectangular coordinates and identify the shape of the curve.

a. Circle of radius 2 centered at a point on the x-axis. Correct Choice

b. Circle of radius 2 centered at a point on the y-axis.

c. Circle of radius 4 centered at a point on the x-axis.

d. Circle of radius 4 centered at a point on the y-axis.

e. Parabola opening to the right.

Solution: $r = 4\cos\theta = 4\frac{x}{r} \quad \Rightarrow \quad r^2 = 4x \quad \Rightarrow \quad x^2 + y^2 - 4x = 0 \quad \Rightarrow \quad (x-2)^2 + y^2 = 4$
This is a circle of radius 2 centered at $(2,0)$.

5. Find the angle between the direction of the line $(x,y,z) = (3+3t, 3-3t, 4)$ and the normal to the plane $2x + 2z = 15$.

a. $90°$

b. $60°$ Correct Choice

c. $45°$

d. $30°$

e. $0°$

Solution: The tangent to the line is $\vec{v} = \langle 3,-3,0\rangle$. The normal to the plane is $\vec{N} = \langle 2,0,2\rangle$.

So $\cos\theta = \dfrac{\vec{v}\bullet\vec{N}}{|\vec{v}||\vec{N}|} = \dfrac{6}{\sqrt{18}\sqrt{8}} = \dfrac{6}{3\sqrt{2}\,2\sqrt{2}} = \dfrac{1}{2}$ and $\theta = 60°$

6. Find the point where the line $(x,y,z) = \vec{r}(t) = (t+2, t-2, 2t-1)$ intersects the plane $3x - y + 2z = 12$. At this point $x + y + z =$

a. 3 Correct Choice

b. 1

c. 0

d. -1

e. -3

Solution: Plug the line into the plane and solve for t:

$$3x - y + 2z = 3(t+2) - (t-2) + 2(2t-1) = 6t + 6 = 12 \qquad \Rightarrow \qquad t = 1$$

So the point is $(x,y,z) = \vec{r}(1) = (3,-1,1)$ and so $x + y + z = 3$.

7. The graph of the equation $x^2 + y^2 - z = -1$ is a

a. Hyperboloid of 1 sheet

b. Hyperboloid of 2 sheets

c. Cone

d. Elliptic Paraboloid opening down

e. Elliptic Paraboloid opening up Correct Choice

Solution: We write it as $z = 1 + x^2 + y^2$.
It is an Elliptic Paraboloid because z is to the first power and x^2 and y^2 have the same sign.
It opens up because $z \geq 1$.

8. Find the equation of the plane thru the point $P = \langle 1,2,3 \rangle$ tangent to the vectors $\vec{a} = \langle 1,2,3 \rangle$ and $\vec{b} = \langle 3,2,1 \rangle$.

a. $-4x - 8y - 4z = 0$

b. $-4x + 8y - 4z = 0$ Correct Choice

c. $-4x - 8y - 4z = 16$

d. $-4x - 8y - 4z = 32$

e. $-4x + 8y - 4z = 32$

Solution: $\vec{N} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \hat{\imath}(2-6) - \hat{\jmath}(1-9) + \hat{k}(2-6) = (-4,8,-4)$

$\vec{N} \cdot X = \vec{N} \cdot P \qquad -4x + 8y - 4z = -4(1) + 8(2) - 4(3) = 0$

9. Find the area of the triangle with adjacent edges $\vec{a} = \langle 1,2,3 \rangle$ and $\vec{b} = \langle 3,2,1 \rangle$.

a. $2\sqrt{2}$

b. $4\sqrt{6}$

c. $2\sqrt{6}$ Correct Choice

d. $\sqrt{6}$

e. $\frac{1}{2}\sqrt{2}$

Solution: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \hat{\imath}(2-6) - \hat{\jmath}(1-9) + \hat{k}(2-6) = (-4,8,-4)$

$A = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{16 + 64 + 16} = \frac{1}{2}\sqrt{96} = 2\sqrt{6}$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (36 points) For the twisted cubic $\vec{r}(t) = \left(2t, \frac{t^3}{3}, t^2\right)$ compute each of the following:

a. (6 pts) The velocity $\vec{v}$

Solution: $\vec{v} = \underline{\langle 2, t^2, 2t \rangle}$

b. (6 pts) The speed $\frac{ds}{dt}$ (Simplify!)

Solution: $\frac{ds}{dt} = |\vec{v}| = \sqrt{4 + t^4 + 4t^2} = \sqrt{(t^2+2)^2} = t^2 + 2$

$\frac{ds}{dt} = \underline{t^2 + 2}$

c. (6 pts) The tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(t^2 + 2) = 2t$

$a_T = \underline{2t}$

d. (6 pts) The mass of a wire in the shape of this twisted cubic between $(0,0,0)$ and $(6,9,9)$ if the linear mass density is $\delta = xz$.

Solution: $|\vec{v}| = t^2 + 2 \qquad \delta = xz = 2t\,t^2 = 2t^3 \qquad (0,0,0) = \vec{r}(0) \qquad (6,9,9) = \vec{r}(3)$

$$M = \int_{(0,0,0)}^{(6,9,9)} \delta\, ds = \int_0^3 xz|\vec{v}|\,dt = \int_0^3 2t^3(t^2+2)\,dt = \left[\frac{t^6}{3} + t^4\right]_0^3 = 3^5 + 3^4 = 243 + 81 = 324$$

$M = \underline{324}$

e. (6 pts) The z-component of the center of mass of the wire between $(0,0,0)$ and $(6,9,9)$ if the linear mass density is $\delta = xz$.

Solution: $$M_z = \int_{(0,0,0)}^{(6,9,9)} z\delta\, ds = \int_0^3 zxz|\vec{v}|\,dt = \int_0^3 t^2 2t^3(t^2+2)\,dt = \left[\frac{t^8}{4} + \frac{4t^6}{6}\right]_0^3$$

$$= \frac{3^8}{4} + 2 \cdot 3^5 = \frac{3^5}{4}(3^3 + 8) = \frac{3^5 \cdot 35}{4}$$

$$\bar{z} = \frac{M_z}{M} = \frac{3^5 \cdot 35}{4 \cdot 324} = \frac{105}{16}$$

$\bar{z} = \underline{\frac{105}{16}}$

f. (6 pts) The work done to move a bead along of a wire in the shape of this twisted cubic between $(0,0,0)$ and $(6,9,9)$ by the force $\vec{F} = (3y, 4x, z)$.

Solution: $\vec{F}(\vec{r}(t)) = (3y, 4x, z) = (t^3, 8t, t^2) \qquad \vec{v} = (2, t^2, 2t)$

$\vec{F} \cdot \vec{v} = 2t^3 + 8t^3 + 2t^3 = 12t^3$

$$W = \int_{(0,0,0)}^{(9,6,9)} \vec{F} \cdot d\vec{s} = \int_0^3 \vec{F} \cdot \vec{v}\,dt = \int_0^3 12t^3\,dt = \left[3t^4\right]_0^3 = 3 \cdot 81 = 243$$

$W = \underline{243}$

11. (15 points) Write the vector $\vec{a} = \langle 2,6,2 \rangle$ as the sum of two vectors $\vec{b}$ and $\vec{c}$ with $\vec{b}$ parallel to $\vec{d} = \langle 1,2,-1 \rangle$ and $\vec{c}$ perpendicular to $\vec{d}$.
Check $\vec{c}$ is perpendicular to $\vec{d}$.

Solution: $\vec{b} = proj_{\vec{d}}\vec{a} = \frac{\vec{a} \cdot \vec{d}}{|\vec{d}|^2}\vec{d} = \frac{2+12-2}{1+4+1}\langle 1,2,-1 \rangle = 2\langle 1,2,-1 \rangle = \langle 2,4,-2 \rangle$

$\vec{c} = \vec{a} - \vec{b} = \langle 2,6,2 \rangle - \langle 2,4,-2 \rangle = \langle 0,2,4 \rangle$

Check: $\vec{d} \cdot \vec{c} = \langle 1,2,-1 \rangle \cdot \langle 0,2,4 \rangle = 4 - 4 = 0$

$\vec{a} = \underbrace{\langle 2,4,-2 \rangle}_{\vec{b}} + \underbrace{\langle 0,2,4 \rangle}_{\vec{c}}$