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MATH 251 Exam 1 Version H Fall 2018

Sections 200/202 Solutions P. Yasskin

1-9	/54	11	/16
10	/36	Total	/106

Multiple Choice: (6 points each. No part credit.)

1. The Galactic Federation is trying to keep a stasis pod stationary in intergalactic space where there is no gravity. They already have 2 tractor beams pulling on the pod with the forces

$$\vec{F}_1 = \langle 4,1,-3 \rangle \quad \text{and} \quad \vec{F}_2 = \langle -2,2,1 \rangle$$

If they apply a 3^{rd} tractor beam on the pod, what should its force $\vec{F}_3$ be to keep the pod stationary?

a. $\vec{F}_3 = \langle 2,3,-2 \rangle$

b. $\vec{F}_3 = \langle 2,-3,-2 \rangle$

c. $\vec{F}_3 = \langle -2,3,2 \rangle$

d. $\vec{F}_3 = \langle -2,-3,2 \rangle$ Correct Choice

e. $\vec{F}_3 = \langle 2,3,2 \rangle$

Solution: To balance forces, $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$. So

$$\vec{F}_3 = -\vec{F}_1 - \vec{F}_2 = -\langle 4,1,-3 \rangle - \langle -2,2,1 \rangle = \langle -2,-3,2 \rangle$$

2. The Galactic Federation moves a stasis pod from $(2,3,4)$ to $(6,9,0)$ by applying the 2 forces:

$$\vec{F}_1 = \langle 4,1,-3 \rangle \quad \text{and} \quad \vec{F}_2 = \langle -2,2,1 \rangle$$

How much work is done by the force $\vec{F}_1$ only?

a. $W = 34$ Correct Choice

b. $W = 33$

c. $W = 22$

d. $W = 10$

e. $W = 0$

Solution: The displacement is $\vec{D} = (6,9,0) - (2,3,4) = \langle 4,6,-4 \rangle$. So the work is

$$W = \vec{F}_1 \cdot \vec{D} = 16 + 6 + 12 = 34$$

3. If a satelite travels from West to East with constant speed in a great circle directly above the Equator of the Earth, where does the unit binormal $\hat{B}$ point?

a. North Correct Choice

b. South

c. West

d. Up

e. Down

Solution: The velocity and $\hat{T}$ point East. The acceleration and $\hat{N}$ point Down toward the center of the Earth. So $\hat{B} = \hat{T} \times \hat{N}$ point North.

4. Convert the polar equation $r = \dfrac{\cos\theta}{\sin^2\theta}$ to rectangular coordinates and identify the shape of the curve.

a. Circle of radius 4 centered at a point on the x-axis.

b. Circle of radius 4 centered at a point on the y-axis.

c. Circle of radius 2 centered at a point on the x-axis.

d. Circle of radius 2 centered at a point on the y-axis.

e. Parabola opening to the right. Correct Choice

Solution: $r = \dfrac{\cos\theta}{\sin^2\theta} = \dfrac{x/r}{y^2/r^2} \quad\Rightarrow\quad \dfrac{y^2}{r} = \dfrac{x}{r} \quad\Rightarrow\quad x = y^2$ Parabola opening to the right.

5. Find the angle between the direction of the line $(x,y,z) = (3+t, 3-t, 4)$ and the normal to the plane $2x - y + z = 7$.

a. $0°$

b. $30°$ Correct Choice

c. $45°$

d. $60°$

e. $90°$

Solution: The tangent to the line is $\vec{v} = \langle 1,-1,0\rangle$. the normal to the plane is $\vec{N} = \langle 2,-1,1\rangle$. So $\cos\theta = \dfrac{\vec{v}\cdot\vec{N}}{|\vec{v}||\vec{N}|} = \dfrac{3}{\sqrt{2}\sqrt{6}} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$ and $\theta = 30°$

6. Find the point where the line $(x,y,z) = \vec{r}(t) = (t+2, t-2, 2t-1)$ intersects the plane $3x - y + 2z = 12$. At this point $x + y + z =$

a. -3

b. -1

c. 0

d. 1

e. 3 Correct Choice

Solution: Plug the line into the plane and solve for t:

$$3x - y + 2z = 3(t+2) - (t-2) + 2(2t-1) = 6t + 6 = 12 \qquad\Rightarrow\qquad t = 1$$

So the point is $(x,y,z) = \vec{r}(1) = (3,-1,1)$ and so $x + y + z = 3$.

7. Is the permutation $p = (2,4,5,6,1,3)$ odd or even and find its inverse $\bar{p}$.

a. Odd $\bar{p} = (3,1,6,5,4,2)$

b. Odd $\bar{p} = (4,3,2,6,1,5)$

c. Odd $\bar{p} = (5,1,6,2,3,4)$ Correct Choice

d. Even $\bar{p} = (4,3,2,6,1,5)$

e. Even $\bar{p} = (5,1,6,2,3,4)$

Solution:
$(2,4,5,6,1,3) \to (1,4,5,6,2,3) \to (1,2,5,6,4,3) \to (1,2,3,6,4,5) \to (1,2,3,4,6,5) \to (1,2,3,4,5,6)$

5 transpositions. Odd. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 6 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 5 & 1 & 6 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ $\bar{p} = (5,1,6,2,3,4)$

8. Find the equation of the hyperplane in $\mathbb{R}^4$ thru the point $P = (1,2,3,5)$ tangent to the vectors $\vec{a} = \langle 1,0,1,0 \rangle$, $\vec{b} = \langle 0,1,0,1 \rangle$ and $\vec{c} = \langle 1,1,0,0 \rangle$. Let the general point be $X = (x,y,z,w)$. (Show your work. I may give part credit.)

a. $x - y - z + w = 1$ Correct Choice

b. $x - y - z + w = -1$

c. $x + y - z - w = -4$

d. $x + y - z - w = 4$

e. $x + y + z + w = 11$

Solution: $\vec{N} = \perp\left(\vec{a},\vec{b},\vec{c}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{l} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - \hat{l}\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$$= \hat{\imath}\left(-1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right) - \hat{\jmath}\left(-1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\right) + \hat{k}\left(\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right) - \hat{l}\left(\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\right)$$

$$= \hat{\imath}(-1(-1)) - \hat{\jmath}(-1(-1)) + \hat{k}(-1) - \hat{l}(-1) = \hat{\imath} - \hat{\jmath} - \hat{k} + \hat{l} = \langle 1,-1,-1,1 \rangle$$

$\vec{N} \cdot X = \vec{N} \cdot P$ $\quad x - y - z + w = 1 - 2 - 3 + 5 = 1$

9. Find the volume of the parallepiped in $\mathbb{R}^4$ with adjacent edges $\vec{a} = \langle 1,0,1,0 \rangle$, $\vec{b} = \langle 0,1,0,1 \rangle$ and $\vec{c} = \langle 1,1,0,0 \rangle$. (Show your work. I may give part credit.)

a. 1

b. $\sqrt{2}$

c. 2 Correct Choice

d. $2\sqrt{2}$

e. 4

Solution: From the solution of the previous problem, $\perp\left(\vec{a},\vec{b},\vec{c}\right) = \hat{\imath} - \hat{\jmath} - \hat{k} + \hat{l}$. So

$$V = \left|\perp\left(\vec{a},\vec{b},\vec{c}\right)\right| = \sqrt{1+1+1+1} = 2$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (36 points) For the twisted cubic $\vec{r}(t) = \left(\frac{t^3}{3}, t^2, 2t\right)$ compute each of the following:

a. (6 pts) The velocity $\vec{v}$

Solution:

$\vec{v} = \underline{\quad (t^2, 2t, 2) \quad}$

b. (6 pts) The speed $\frac{ds}{dt}$ (Simplify!)

Solution: $\frac{ds}{dt} = |\vec{v}| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$

$\frac{ds}{dt} = \underline{\quad t^2 + 2 \quad}$

c. (6 pts) The tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(t^2 + 2) = 2t$

$a_T = \underline{\quad 2t \quad}$

d. (6 pts) The mass of a wire in the shape of this twisted cubic between $(0,0,0)$ and $(9,9,6)$ if the linear mass density is $\delta = yz$.

Solution: $|\vec{v}| = t^2 + 2$ $\quad \delta = yz = t^2\, 2t = 2t^3$ $\quad (0,0,0) = \vec{r}(0)$ $\quad (9,9,6) = \vec{r}(3)$

$$M = \int_{(0,0,0)}^{(9,9,6)} \delta\, ds = \int_0^3 yz|\vec{v}|\, dt = \int_0^3 2t^3(t^2+2)\, dt = \left[\frac{t^6}{3} + t^4\right]_0^3 = 3^5 + 3^4 = 243 + 81 = 324$$

$M = \underline{\quad 324 \quad}$

e. (6 pts) The y-component of the center of mass of the wire between $(0,0,0)$ and $(9,9,6)$ if the linear mass density is $\delta = yz$.

Solution:

$$M_y = \int_{(0,0,0)}^{(9,9,6)} y\,\delta\, ds = \int_0^3 yyz|\vec{v}|\, dt = \int_0^3 t^2 2t^3(t^2+2)\, dt = \left[\frac{t^8}{4} + \frac{4t^6}{6}\right]_0^3$$

$$= \frac{3^8}{4} + 2 \cdot 3^5 = \frac{3^5}{4}(3^3 + 8) = \frac{3^5 \cdot 35}{4}$$

$$\bar{y} = \frac{M_y}{M} = \frac{3^5 \cdot 35}{4 \cdot 324} = \frac{105}{16}$$

$\bar{y} = \underline{\quad \frac{105}{16} \quad}$

f. (6 pts) The work done to move a bead along of a wire in the shape of this twisted cubic between $(0,0,0)$ and $(9,9,6)$ by the force $\vec{F} = (z, 2y, -3x)$.

Solution: $\vec{F}(\vec{r}(t)) = (z, 2y, -3x) = (2t, 2t^2, -t^3)$ $\quad \vec{v} = (t^2, 2t, 2)$

$\vec{F} \cdot \vec{v} = 2t^3 + 4t^3 - 2t^3 = 4t^3$

$$W = \int_{(0,0,0)}^{(9,9,6)} \vec{F} \cdot d\vec{s} = \int_0^3 \vec{F} \cdot \vec{v}\, dt = \int_0^3 8t^3\, dt = \left[t^4\right]_0^3 = 81$$

$W = \underline{\quad 81 \quad}$

11. (15 points) Write the vector $\vec{a} = \langle 2,2,6\rangle$ as the sum of two vectors $\vec{b}$ and $\vec{c}$ with $\vec{b}$ parallel to $\vec{d} = \langle 1,-1,2\rangle$ and $\vec{c}$ perpendicular to $\vec{d}$.
Check $\vec{c}$ is perpendicular to $\vec{d}$.

Solution: $\vec{b} = proj_{\vec{d}}\vec{a} = \dfrac{\vec{a}\cdot\vec{d}}{|\vec{d}|^2}\vec{d} = \dfrac{2-2+12}{1+1+4}\langle 1,-1,2\rangle = 2\langle 1,-1,2\rangle = \langle 2,-2,4\rangle$

$\vec{c} = \vec{a} - \vec{b} = \langle 2,2,6\rangle - \langle 2,-2,4\rangle = \langle 0,4,2\rangle$

Check: $\vec{d}\cdot\vec{c} = \langle 1,-1,2\rangle \cdot \langle 0,4,2\rangle = -4+4 = 0$

$\vec{a} = \underset{\vec{b}}{\underline{\langle 2,-2,4\rangle}} + \underset{\vec{c}}{\underline{\langle 0,4,2\rangle}}$