Name________________________

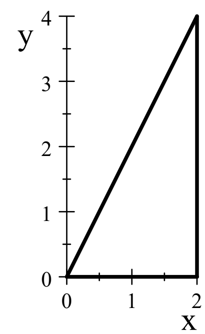
MATH 251 Exam 3 Version A Fall 2018

Sections 504/505 Solutions P. Yasskin

Multiple Choice: (7 points each. No part credit.)

1-10	/70	12	/20
11	/20	Total	/110

1. Find the mass of a triangular plate with vertices $(0,0)$, $(2,0)$ and $(2,4)$ if the density is $\delta = x$.



a. $M = \frac{16}{3}$ Correct Choice

b. $M = \frac{8}{3}$

c. $M = 8$

d. $M = 4$

e. $M = 2$

Solution: $0 \le x \le 2$ $\quad 0 \le y \le 2x$ Type x integral:

$$M = \iint \delta\, dA = \int_0^2 \int_0^{2x} x\, dy\, dx = \int_0^2 \left[xy\right]_{y=0}^{2x} dx = \int_0^2 2x^2\, dx = \left[\frac{2x^3}{3}\right]_0^2 = \frac{16}{3}$$

2. Find the x-component of the center of mass of a triangular plate with vertices $(0,0)$, $(2,0)$ and $(2,4)$ if the density is $\delta = x$.

a. $\bar{x} = 2$

b. $\bar{x} = 4$

c. $\bar{x} = 8$

d. $\bar{x} = \frac{3}{2}$ Correct Choice

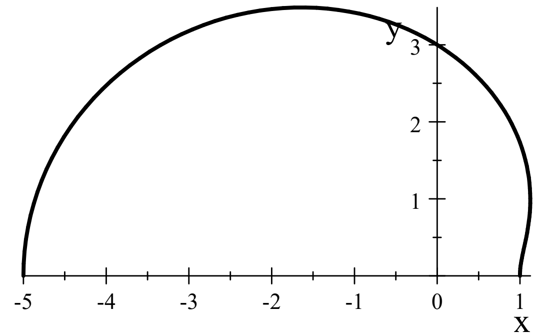
e. $\bar{x} = \frac{2}{3}$

Solution: We continue the solution to problem 1. The x-moment is:

$$M_x = \iint x\delta\, dA = \int_0^2 \int_0^{2x} x^2\, dy\, dx = \int_0^2 \left[x^2 y\right]_{y=0}^{2x} dx = \int_0^2 2x^3\, dx = \left[\frac{2x^4}{4}\right]_0^2 = 8$$

So: $\bar{x} = \frac{M_x}{M} = \frac{8}{1}\frac{3}{16} = \frac{3}{2}$

3. Find the area of the upper half of the limacon $r = 3 - 2\cos\theta$.



a. $A = \frac{9\pi}{2}$

b. $A = \frac{11\pi}{2}$ Correct Choice

c. $A = \frac{13\pi}{2}$

d. $A = 9\pi$

e. $A = 11\pi$

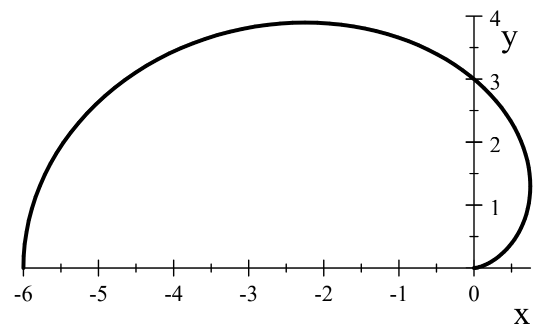
Solution: We use polar coordinates $0 \le \theta \le \pi \quad 0 \le r \le 3 - 2\cos\theta$

$$A = \iint 1\,dA = \int_0^\pi \int_0^{3-2\cos\theta} r\,dr\,d\theta = \int_0^\pi \left[\frac{r^2}{2}\right]_{r=0}^{3-2\cos\theta} d\theta = \frac{1}{2}\int_0^\pi (3-2\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^\pi (9 - 12\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^\pi (9 - 12\cos\theta + 2(1+\cos 2\theta))\,d\theta = \frac{1}{2}\left[9\theta - 12\sin\theta + 2\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^\pi = \frac{11\pi}{2}$$

4. Given: The area of the upper half of the cardioid $r = 3 - 3\cos\theta$ is $A = \frac{27}{4}\pi$.

Find the y-component of its centroid.



a. $\bar{y} = 12$

b. $\bar{y} = 36$

c. $\bar{y} = \frac{16}{9\pi}$

d. $\bar{y} = \frac{3\pi}{16}$

e. $\bar{y} = \frac{16}{3\pi}$ Correct Choice

Solution: The area is $A = \iint 1\,dA = \int_0^\pi \int_0^{3-3\cos\theta} r\,dr\,d\theta = \frac{27}{4}\pi$

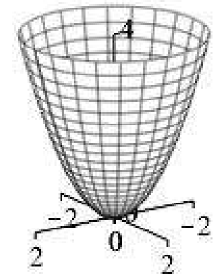
You did not need to compute this, but the form of the integral is useful.
The y-moment of the area is:

$$A_y = \iint y\,dA = \int_0^\pi \int_0^{3-3\cos\theta} (r\sin\theta) r\,dr\,d\theta = \int_0^\pi \left[\frac{r^3}{3}\right]_{r=0}^{3-3\cos\theta} \sin\theta\,d\theta = \frac{1}{3}\int_0^\pi (3-3\cos\theta)^3 \sin\theta\,d\theta$$

Let $u = 3 - 3\cos\theta$. Then $du = 3\sin\theta\,d\theta$ and

$$A_y = \frac{1}{9}\int_0^6 u^3\,du = \frac{1}{9}\left[\frac{u^4}{4}\right]_0^6 = \frac{6^4}{36} = 36 \qquad \text{So } \bar{y} = \frac{A_y}{A} = \frac{36}{1}\frac{4}{27\pi} = \frac{16}{3\pi}$$

5. Given: The solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ has volume $V = 8\pi$. Find the z-component of its centroid.



a. $\bar{y} = \frac{16}{5}$

b. $\bar{y} = \frac{8}{5}$

c. $\bar{y} = \frac{8}{3}$ Correct Choice

d. $\bar{y} = \frac{64\pi}{3}$

e. $\bar{y} = \frac{3}{64\pi}$

Solution: In cylindrical coordinates, the volume is

$$V = \iiint 1\, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r\, dz\, dr\, d\theta = 2\pi \int_0^2 \left[rz \right]_{z=r^2}^4 dr = 2\pi \int_0^2 (4r - r^3)\, dr = 2\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2 = 8\pi$$

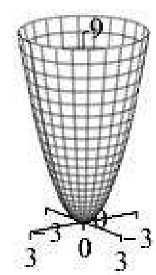
You did not need to compute this, but the form of the integral is useful. The z-moment is:

$$V_z = \iiint z\, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 zr\, dz\, dr\, d\theta = 2\pi \int_0^2 \left[\frac{z^2}{2} r \right]_{z=r^2}^4 dr = 2\pi \int_0^2 \left(8r - \frac{r^5}{2} \right) dr = 2\pi \left[4r^2 - \frac{r^6}{12} \right]_0^2$$

$$= 2\pi \left(16 - \frac{16}{3} \right) = \frac{64\pi}{3} \qquad \text{So } \bar{z} = \frac{V_z}{V} = \frac{64\pi}{3} \frac{1}{8\pi} = \frac{8}{3}$$

6. Given: The solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 9$ has centroid $(0,0,6)$. If the temperature of the solid is $T = 4 + z$ find the average temperature.

HINT: You don't need to compute any integral.



a. $T_{\text{ave}} = 4$

b. $T_{\text{ave}} = 7$

c. $T_{\text{ave}} = \frac{17}{2}$

d. $T_{\text{ave}} = 10$ Correct Choice

e. $T_{\text{ave}} = 13$

Solution: $T_{\text{ave}} = \dfrac{\iiint T\, dV}{V} = \dfrac{\iiint 4 + z\, dV}{\iiint 1\, dV} = \dfrac{4 \iiint 1\, dV + \iiint z\, dV}{\iiint 1\, dV} = 4 + \bar{z} = 4 + 6 = 10$

7. Find the volume of an ice cream cone between the cone $z = \sqrt{x^2+y^2}$ and the upper piece of the sphere $x^2+y^2+z^2 = 9$.

a. $18\pi\left(1-\frac{1}{\sqrt{2}}\right)$ Correct Choice

b. $9\pi\left(1-\frac{1}{\sqrt{2}}\right)$

c. $\frac{18\pi}{\sqrt{2}}$

d. $\frac{9\pi}{\sqrt{2}}$

e. $\frac{9}{2}\pi^2$

Solution: In spherical coordinates, $z = \rho\cos\varphi$ and $\sqrt{x^2+y^2} = r = \rho\sin\varphi$. So the cone is $\rho\cos\varphi = \rho\sin\varphi$ or $\tan\varphi = 1$ or $\varphi = \frac{\pi}{4}$. Further, the sphere is $\rho = 3$. So the volume is

$$V = \iiint 1\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\left[-\cos\varphi\right]_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^3 = 2\pi\left(-\frac{1}{\sqrt{2}} - -1\right)9 = 18\pi\left(1-\frac{1}{\sqrt{2}}\right)$$

8. Compute $\int_0^1\int_x^1 xe^{y^3}\,dy\,dx$.

HINT: Reverse the order of integration.

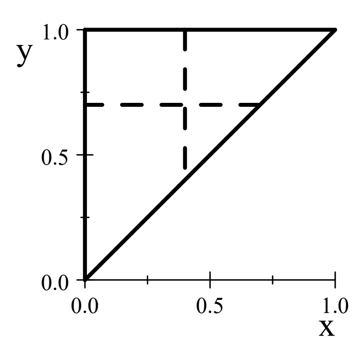
a. $\frac{e}{6}$

b. $\frac{e}{6}-\frac{1}{6}$ Correct Choice

c. $\frac{3e}{2}$

d. $\frac{3e}{2}-\frac{3}{2}$

Solution: Here is a plot of the region:



Original: $0 \le x \le 1 \quad x \le y \le 1$

Reversed: $0 \le y \le 1 \quad 0 \le x \le y$

$$\int_0^1\int_x^1 xe^{y^3}\,dy\,dx = \int_0^1\int_0^y xe^{y^3}\,dx\,dy$$

$$= \int_0^1\left[\frac{x^2}{2}\right]_{x=0}^{y} e^{y^3}\,dy = \frac{1}{2}\int_0^1 y^2e^{y^3}\,dy$$

$$u = y^3 \qquad du = 3y^2\,dy \qquad \frac{1}{3}du = y^2\,dy$$

$$\int_0^1\int_x^1 xe^{y^3}\,dy\,dx = \frac{1}{6}\int_0^1 e^u\,du = \frac{1}{6}e^u\Big|_0^1 = \frac{e}{6}-\frac{1}{6}$$

9. Find the work done to push a bead along a wire in the shape of the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ from $(1,1,1)$ to $(2,4,8)$ if the force is $\vec{F} = \langle z, 2y, x \rangle$.

a. 56

b. 45

c. 30 Correct Choice

d. $\frac{45}{2}$

e. 15

Solution: The velocity is $\vec{v} = \langle 1, 2t, 3t^2 \rangle$. On the curve, the force is $\vec{F} = \langle t^3, 2t^2, t \rangle$.
So $\vec{F} \cdot \vec{v} = t^3 + 4t^3 + 3t^3 = 8t^3$. So the work is
$W = \int_{(1,1,1)}^{(2,4,8)} \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F} \cdot \vec{v}\, dt = \int_1^2 8t^3\, dt = \left[2t^4\right]_1^2 = 30$

10. Find the mass of the conical **surface** $z = \sqrt{x^2 + y^2}$ for $z \le 4$ if the surface density is $\delta = z\sqrt{x^2 + y^2}$. The surface may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.
SUGGESTION: Do problem 11 first.

a. $M = 8\sqrt{2}\,\pi$

b. $M = 64\pi$

c. $M = 128\pi$

d. $M = 64\sqrt{2}\,\pi$

e. $M = 128\sqrt{2}\,\pi$ Correct Choice

Solution: On the surface, $z = r$ and $\sqrt{x^2 + y^2} = r$. So $\delta = z\sqrt{x^2 + y^2} = r^2$.
From problem 11, $\left|\vec{N}\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = r\sqrt{2}$.
The surface is $z = r$ and $z \le 4$ So $r \le 4$.
So the mass is: $M = \iint_S \delta\, dS = \int\int \delta\left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^4 r^2 r\sqrt{2}\, dr\, d\theta = 2\pi\sqrt{2}\left[\frac{r^4}{4}\right]_0^4 = 128\sqrt{2}\,\pi$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (20 points) Find the flux $\iint \vec{F} \cdot d\vec{S}$ of the vector field $\vec{F} = \langle 6xz^2, 6yz^2, z^3 \rangle$ down and out through the conical **surface** $z = \sqrt{x^2 + y^2}$ for $z \le 4$. Follow these steps:

Paramertize the surface as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.

a. Compute the tangent vectors:

$$\begin{array}{lcccc} & \hat{\imath} & \hat{\jmath} & \hat{k} & \\ \vec{e}_r = \dfrac{\partial \vec{R}}{\partial r} = \langle & \cos\theta, & \sin\theta, & 1 & \rangle \\ \vec{e}_\theta = \dfrac{\partial \vec{R}}{\partial \theta} = \langle & -r\sin\theta, & r\cos\theta, & 0 & \rangle \end{array}$$

b. Compute the normal vector and check, explain and fix the orientation:

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(0 - r\cos\theta) - \hat{\jmath}(- -r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta)$$

$$\vec{N} = \langle \quad -r\cos\theta \quad , \quad -r\sin\theta \quad , \quad r \quad \rangle$$

This is in (x and y are negative in the first quadrant) and up (z is positive). We want down and out. So we reverse it:

$$\vec{N} = \langle \quad r\cos\theta \quad , \quad r\sin\theta \quad , \quad -r \quad \rangle$$

c. Evaluate the vector field on the surface:

$$\vec{F}\left(\vec{R}(r,\theta)\right) = \langle \quad 6r^3\cos\theta \quad , \quad 6r^3\sin\theta \quad , \quad r^3 \quad \rangle$$

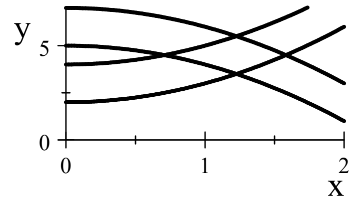
d. Compute the dot product:

$$\vec{F} \cdot \vec{N} = 6r^4\cos^2\theta + 6r^4\sin^2\theta - r^4 = 5r^4$$

e. Compute the flux integral:

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^4 5r^4\, dr\, d\theta = 2\pi [r^5]_0^4 = 2 \cdot 4^5 \pi = 2048\pi$$

12. (20 points) Compute the integral $\iint xy\,dA$ over the region in the first quadrant bounded by $y = 2 + x^2$, $y = 4 + x^2$, $y = 5 - x^2$, and $y = 7 - x^2$.



a. Define the curvilinear coordinates u and v by $y = u + x^2$ and $y = v - x^2$.

What are the 4 boundaries in terms of u and v?

$u = 2 \qquad u = 4 \qquad v = 5 \qquad v = 7$

b. Solve for x and y in terms of u and v. Express the results as a position vector.

Add: $\quad 2y = u + x^2 + v - x^2 = u + v \qquad y = \dfrac{u+v}{2}$

Subtract: $\quad 0 = u + x^2 - v + x^2 = u - v + 2x^2 \qquad 2x^2 = v - u \qquad x = \dfrac{\sqrt{v-u}}{\sqrt{2}}$

$$\vec{R}(u,v) = (x(u,v), y(u,v)) = \left(\frac{\sqrt{v-u}}{\sqrt{2}}, \frac{u+v}{2}\right)$$

c. Find the coordinate tangent vectors:

$$\vec{e}_u = \frac{\partial \vec{R}}{\partial u} = \left(\frac{1}{2\sqrt{2}}\frac{-1}{\sqrt{v-u}}, \quad \frac{1}{2}\right)$$

$$\vec{e}_v = \frac{\partial \vec{R}}{\partial v} = \left(\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{v-u}}, \quad \frac{1}{2}\right)$$

d. Compute the Jacobian determinant:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{2}}\frac{-1}{\sqrt{v-u}} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{v-u}} & \frac{1}{2} \end{vmatrix} = \frac{1}{4\sqrt{2}}\frac{-1}{\sqrt{v-u}} - \frac{1}{4\sqrt{2}}\frac{1}{\sqrt{v-u}} = \frac{1}{2\sqrt{2}}\frac{-1}{\sqrt{v-u}}$$

e. Compute the Jacobian factor:

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2\sqrt{2}\sqrt{v-u}}$$

f. Compute the integrand:

$$xy = \frac{\sqrt{v-u}}{\sqrt{2}} \bullet \frac{u+v}{2} = \frac{(u+v)\sqrt{v-u}}{2\sqrt{2}}$$

g. Compute the integral:

$$\iint xy\,dA = \int_5^7\int_2^4 \frac{(u+v)\sqrt{v-u}}{2\sqrt{2}} \frac{1}{2\sqrt{2}\sqrt{v-u}}\,du\,dv = \frac{1}{8}\int_5^7\int_2^4 (u+v)\,du\,dv$$

$$= \frac{1}{8}\int_5^7\left[\frac{u^2}{2} + uv\right]_{u=2}^4 dv = \frac{1}{8}\int_5^7 (6+2v)\,dv = \frac{1}{8}[6v + v^2]_{v=5}^7 = \frac{1}{8}(42 + 49 - 30 - 25) = \frac{9}{2}$$