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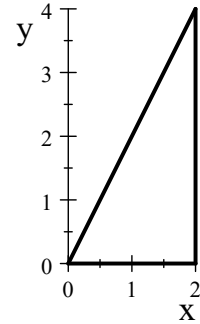
MATH 251 Exam 3 Version H Fall 2018
Sections 504/505 Solutions P. Yasskin

1-10	/70	12	/20
11	/20	Total	/110

Multiple Choice: (7 points each. No part credit.)

1. Find the mass of a triangular plate with vertices $(0,0)$, $(2,0)$ and $(2,4)$ if the density is $\delta = x$.

- a. $M = 2$
- b. $M = 4$
- c. $M = 8$
- d. $M = \frac{8}{3}$
- e. $M = \frac{16}{3}$ Correct Choice



Solution: $0 \leq x \leq 2$ $0 \leq y \leq 2x$ Type x integral:

$$M = \iint \delta dA = \int_0^2 \int_0^{2x} x dy dx = \int_0^2 [xy]_{y=0}^{2x} dx = \int_0^2 2x^2 dx = \left[\frac{2x^3}{3} \right]_0^2 = \frac{16}{3}$$

2. Find the x -component of the center of mass of a triangular plate with vertices $(0,0)$, $(2,0)$ and $(2,4)$ if the density is $\delta = x$.

- a. $\bar{x} = 2$
- b. $\bar{x} = 4$
- c. $\bar{x} = 8$
- d. $\bar{x} = \frac{3}{2}$ Correct Choice
- e. $\bar{x} = \frac{2}{3}$

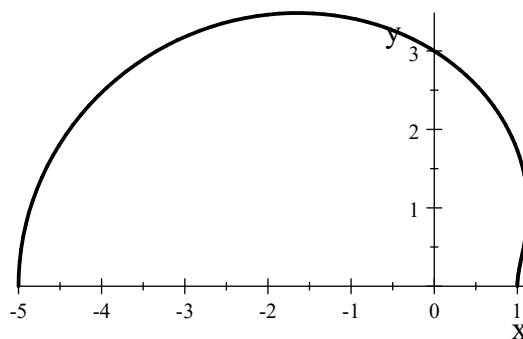
Solution: We continue the solution to problem 1. The x -moment is:

$$M_x = \iint x\delta dA = \int_0^2 \int_0^{2x} x^2 dy dx = \int_0^2 [x^2y]_{y=0}^{2x} dx = \int_0^2 2x^3 dx = \left[\frac{2x^4}{4} \right]_0^2 = 8$$

$$\text{So: } \bar{x} = \frac{M_x}{M} = \frac{8}{\frac{16}{3}} = \frac{3}{2}$$

3. Find the area of the upper half of the limaçon $r = 3 - 2 \cos \theta$.

- a. $A = \frac{9\pi}{2}$
 b. $A = \frac{11\pi}{2}$ Correct Choice
 c. $A = \frac{13\pi}{2}$
 d. $A = 9\pi$
 e. $A = 11\pi$



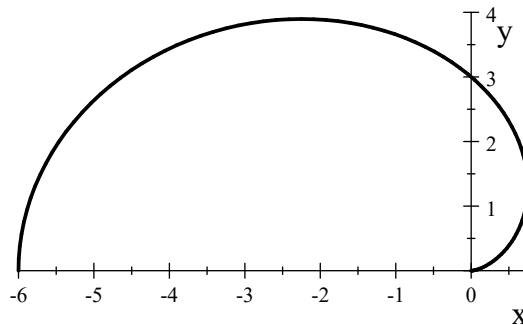
Solution: We use polar coordinates $0 \leq \theta \leq \pi$ $0 \leq r \leq 3 - 2 \cos \theta$

$$A = \iint 1 dA = \int_0^\pi \int_0^{3-2\cos\theta} r dr d\theta = \int_0^\pi \left[\frac{r^2}{2} \right]_{r=0}^{3-2\cos\theta} d\theta = \frac{1}{2} \int_0^\pi (3 - 2 \cos \theta)^2 d\theta = \frac{1}{2} \int_0^\pi (9 - 12 \cos \theta + 4 \cos^2 \theta) d\theta$$

$$= \frac{1}{2} \int_0^\pi (9 - 12 \cos \theta + 2(1 + \cos 2\theta)) d\theta = \frac{1}{2} \left[9\theta - 12 \sin \theta + 2 \left(\theta + \frac{\sin 2\theta}{2} \right) \right]_0^\pi = \frac{11\pi}{2}$$

4. Given: The area of the upper half of the cardioid $r = 3 - 3 \cos \theta$ is $A = \frac{27}{4}\pi$. Find the y -component of its centroid.

- a. $\bar{y} = \frac{16}{3\pi}$ Correct Choice
 b. $\bar{y} = \frac{3\pi}{16}$
 c. $\bar{y} = \frac{16}{9\pi}$
 d. $\bar{y} = 36$
 e. $\bar{y} = 12$



Solution: The area is $A = \iint 1 dA = \int_0^\pi \int_0^{3-3\cos\theta} r dr d\theta = \frac{27}{4}\pi$

You did not need to compute this, but the form of the integral is useful.

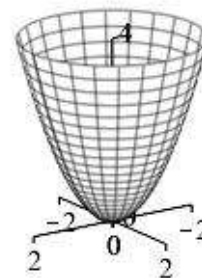
The y -moment of the area is:

$$A_y = \iint y dA = \int_0^\pi \int_0^{3-3\cos\theta} (r \sin \theta) r dr d\theta = \int_0^\pi \left[\frac{r^3}{3} \right]_{r=0}^{3-3\cos\theta} \sin \theta d\theta = \frac{1}{3} \int_0^\pi (3 - 3 \cos \theta)^3 \sin \theta d\theta$$

Let $u = 3 - 3 \cos \theta$. Then $du = 3 \sin \theta d\theta$ and

$$A_y = \frac{1}{9} \int_0^6 u^3 du = \frac{1}{9} \left[\frac{u^4}{4} \right]_0^6 = \frac{6^4}{36} = 36 \quad \text{So } \bar{y} = \frac{A_y}{A} = \frac{36}{1} \frac{4}{27\pi} = \frac{16}{3\pi}$$

5. Given: The solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ has volume $V = 8\pi$. Find the z -component of its centroid.



- a. $\bar{y} = \frac{3}{64\pi}$
 b. $\bar{y} = \frac{64\pi}{3}$
 c. $\bar{y} = \frac{8}{3}$ Correct Choice
 d. $\bar{y} = \frac{8}{5}$
 e. $\bar{y} = \frac{16}{5}$

Solution: In cylindrical coordinates, the volume is

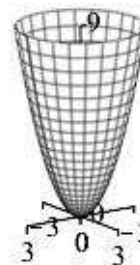
$$V = \iiint 1 dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r dz dr d\theta = 2\pi \int_0^2 [rz]_{z=r^2}^4 dr = 2\pi \int_0^2 (4r - r^3) dr = 2\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2 = 8\pi$$

You did not need to compute this, but the form of the integral is useful. The z -moment is:

$$V_z = \iiint z dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 zr dz dr d\theta = 2\pi \int_0^2 \left[\frac{z^2}{2} r \right]_{z=r^2}^4 dr = 2\pi \int_0^2 \left(8r - \frac{r^5}{2} \right) dr = 2\pi \left[4r^2 - \frac{r^6}{12} \right]_0^2$$

$$= 2\pi \left(16 - \frac{16}{3} \right) = \frac{64\pi}{3} \quad \text{So } \bar{z} = \frac{V_z}{V} = \frac{64\pi}{3} \frac{1}{8\pi} = \frac{8}{3}$$

6. Given: The solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 9$ has centroid $(0, 0, 6)$. If the temperature of the solid is $T = 4 + z$ find the average temperature.



HINT: You don't need to compute any integral.

- a. $T_{\text{ave}} = 4$
 b. $T_{\text{ave}} = 7$
 c. $T_{\text{ave}} = \frac{17}{2}$
 d. $T_{\text{ave}} = 10$ Correct Choice
 e. $T_{\text{ave}} = 13$

Solution: $T_{\text{ave}} = \frac{\iiint T dV}{V} = \frac{\iiint 4 + z dV}{\iiint 1 dV} = \frac{4 \iiint 1 dV + \iiint z dV}{\iiint 1 dV} = 4 + \bar{z} = 4 + 6 = 10$

7. Find the volume of an ice cream cone between the cone $z = \sqrt{x^2 + y^2}$ and the upper piece of the sphere $x^2 + y^2 + z^2 = 9$.

- a. $\frac{9}{2}\pi^2$
- b. $\frac{9\pi}{\sqrt{2}}$
- c. $\frac{18\pi}{\sqrt{2}}$
- d. $9\pi\left(1 - \frac{1}{\sqrt{2}}\right)$
- e. $18\pi\left(1 - \frac{1}{\sqrt{2}}\right)$ Correct Choice

Solution: In spherical coordinates, $z = \rho \cos \phi$ and $\sqrt{x^2 + y^2} = r = \rho \sin \phi$. So the cone is $\rho \cos \phi = \rho \sin \phi$ or $\tan \phi = 1$ or $\phi = \frac{\pi}{4}$. Further, the sphere is $\rho = 3$. So the volume is

$$V = \iiint 1 dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \phi d\rho d\phi d\theta = 2\pi [-\cos \phi]_0^{\pi/4} \left[\frac{\rho^3}{3} \right]_0^3 = 2\pi \left(-\frac{1}{\sqrt{2}} - -1 \right) 9 = 18\pi \left(1 - \frac{1}{\sqrt{2}} \right)$$

8. Compute $\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \frac{1}{(1+x^2+y^2)^2} dy dx$.

HINT: Change coordinates.

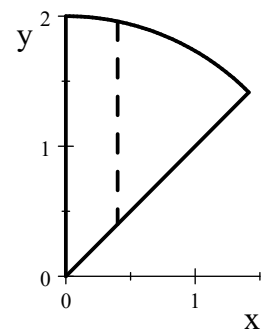
- a. $\frac{\pi}{40}$
- b. $\frac{\pi}{10}$ Correct Choice
- c. $\frac{\pi}{5}$
- d. $\frac{2\pi}{5}$
- e. $\frac{-\pi}{40}$

Solution: Here is a plot of the region:

Original: $0 \leq x \leq \sqrt{2} \quad x \leq y \leq \sqrt{4-x^2}$

Polar: $0 \leq r \leq 2 \quad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned} \int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \frac{1}{(1+x^2+y^2)^2} dy dx &= \int_{\pi/4}^{\pi/2} \int_0^2 \frac{1}{(1+r^2)^2} r dr d\theta \\ &= \left(\frac{\pi}{2} - \frac{\pi}{4} \right) \left[\frac{1}{2} \frac{-1}{1+r^2} \right]_{r=0}^2 = \frac{\pi}{8} \left(\frac{-1}{5} - \frac{-1}{1} \right) \\ &= \frac{\pi}{8} \cdot \frac{4}{5} = \frac{\pi}{10} \end{aligned}$$



9. Find the work done to push a bead along a wire in the shape of the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ from $(1, 1, 1)$ to $(2, 4, 8)$ if the force is $\vec{F} = \langle z, 2y, x \rangle$.
- 56
 - 45
 - 30 Correct Choice
 - $\frac{45}{2}$
 - 15

Solution: The velocity is $\vec{v} = \langle 1, 2t, 3t^2 \rangle$. On the curve, the force is $\vec{F} = \langle t^3, 2t^2, t \rangle$.

So $\vec{F} \cdot \vec{v} = t^3 + 4t^3 + 3t^3 = 8t^3$. So the work is

$$W = \int_{(1,1,1)}^{(2,4,8)} \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F} \cdot \vec{v} dt = \int_1^2 8t^3 dt = [2t^4]_1^2 = 30$$

10. Find the mass of the conical **surface** $z = \sqrt{x^2 + y^2}$ for $z \leq 4$ if the surface density is $\delta = z\sqrt{x^2 + y^2}$. The surface may be parametrized by $\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r)$.

SUGGESTION: Do problem 11 first.

- $M = 128\sqrt{2}\pi$ Correct Choice
- $M = 64\sqrt{2}\pi$
- $M = 128\pi$
- $M = 64\pi$
- $M = 8\sqrt{2}\pi$

Solution: On the surface, $z = r$ and $\sqrt{x^2 + y^2} = r$. So $\delta = z\sqrt{x^2 + y^2} = r^2$.

From problem 11, $|\vec{N}| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = r\sqrt{2}$.

The surface is $z = r$ and $z \leq 4$ So $r \leq 4$.

So the mass is:
$$M = \iint_S \delta dS = \iint \delta |\vec{N}| dr d\theta = \int_0^{2\pi} \int_0^4 r^2 r \sqrt{2} dr d\theta = 2\pi\sqrt{2} \left[\frac{r^4}{4} \right]_0^4 = 128\sqrt{2}\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (20 points) Find the flux $\iint \vec{F} \cdot d\vec{S}$ of the vector field $\vec{F} = \langle 6xz^2, 6yz^2, z^3 \rangle$ down and out through the conical **surface** $z = \sqrt{x^2 + y^2}$ for $z \leq 4$. Follow these steps:

Parametrize the surface as $\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r)$.

- a. Compute the tangent vectors:

$$\vec{e}_r = \frac{\partial \vec{R}}{\partial r} = \langle \quad \hat{i} \quad \cos \theta \quad , \quad \hat{j} \quad \sin \theta \quad , \quad \hat{k} \quad 1 \quad \rangle$$

$$\vec{e}_\theta = \frac{\partial \vec{R}}{\partial \theta} = \langle \quad -r \sin \theta \quad , \quad r \cos \theta \quad , \quad 0 \quad \rangle$$

- b. Compute the normal vector and check, explain and fix the orientation:

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{i}(0 - r \cos \theta) - \hat{j}(-r \sin \theta) + \hat{k}(r \cos^2 \theta - -r \sin^2 \theta)$$

$$\vec{N} = \langle \quad -r \cos \theta \quad , \quad -r \sin \theta \quad , \quad r \quad \rangle$$

This is in (x and y are negative in the first quadrant) and up (z is positive). We want down and out. So we reverse it:

$$\vec{N} = \langle \quad r \cos \theta \quad , \quad r \sin \theta \quad , \quad -r \quad \rangle$$

- c. Evaluate the vector field on the surface:

$$\vec{F}(\vec{R}(r, \theta)) = \langle \quad 6r^3 \cos \theta \quad , \quad 6r^3 \sin \theta \quad , \quad r^3 \quad \rangle$$

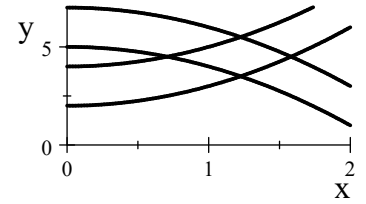
- d. Compute the dot product:

$$\vec{F} \cdot \vec{N} = 6r^4 \cos^2 \theta + 6r^4 \sin^2 \theta - r^4 = 5r^4$$

- e. Compute the flux integral:

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N} dr d\theta = \int_0^{2\pi} \int_0^4 5r^4 dr d\theta = 2\pi [r^5]_0^4 = 2 \cdot 4^5 \pi = 2048\pi$$

12. (20 points) Compute the integral $\iint xy dA$ over the region in the first quadrant bounded by $y = 2 + x^2$, $y = 4 + x^2$, $y = 5 - x^2$, and $y = 7 - x^2$.



- a. Define the curvilinear coordinates u and v by $y = u + x^2$ and $y = v - x^2$.

What are the 4 boundaries in terms of u and v ?

$$u = 2 \quad u = 4 \quad v = 5 \quad v = 7$$

- b. Solve for x and y in terms of u and v . Express the results as a position vector.

$$\text{Add:} \quad 2y = u + x^2 + v - x^2 = u + v \quad y = \frac{u + v}{2}$$

$$\text{Subtract:} \quad 0 = u + x^2 - v + x^2 = u - v + 2x^2 \quad 2x^2 = v - u \quad x = \frac{\sqrt{v - u}}{\sqrt{2}}$$

$$\vec{R}(u, v) = (x(u, v), y(u, v)) = \left(\frac{\sqrt{v - u}}{\sqrt{2}}, \frac{u + v}{2} \right)$$

- c. Find the coordinate tangent vectors:

$$\vec{e}_u = \frac{\partial \vec{R}}{\partial u} = \left(\frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v - u}}, \frac{1}{2} \right)$$

$$\vec{e}_v = \frac{\partial \vec{R}}{\partial v} = \left(\frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v - u}}, \frac{1}{2} \right)$$

- d. Compute the Jacobian determinant:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v - u}} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v - u}} & \frac{1}{2} \end{vmatrix} = \frac{1}{4\sqrt{2}} \frac{-1}{\sqrt{v - u}} - \frac{1}{4\sqrt{2}} \frac{1}{\sqrt{v - u}} = \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v - u}}$$

- e. Compute the Jacobian factor:

$$J = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{2\sqrt{2} \sqrt{v - u}}$$

- f. Compute the integrand:

$$xy = \frac{\sqrt{v - u}}{\sqrt{2}} \cdot \frac{u + v}{2} = \frac{(u + v) \sqrt{v - u}}{2\sqrt{2}}$$

- g. Compute the integral:

$$\iint xy dA = \int_5^7 \int_2^4 \frac{(u + v) \sqrt{v - u}}{2\sqrt{2}} \frac{1}{2\sqrt{2} \sqrt{v - u}} du dv = \frac{1}{8} \int_5^7 \int_2^4 (u + v) du dv$$

$$= \frac{1}{8} \int_5^7 \left[\frac{u^2}{2} + uv \right]_{u=2}^4 dv = \frac{1}{8} \int_5^7 (6 + 2v) dv = \frac{1}{8} [6v + v^2]_{v=5}^7 = \frac{1}{8} (42 + 49 - 30 - 25) = \frac{9}{2}$$