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MATH 251 Final Version A Fall 2018

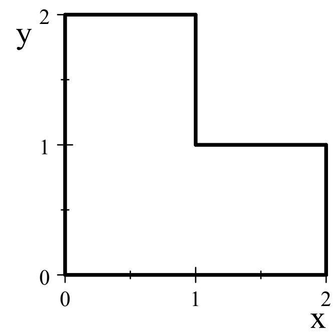
Sections 505 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1-10	/60	12	/20
11	/ 5	13	/25
		Total	/110

1. Compute $I = \int_{\partial R} (3y + 2xy^3)\,dx + (8x + 3x^2y^2)\,dy$

over the complete boundary of the L-shape shown at the right traversed counterclockwise.
HINT: Use a theorem.



a. $I = 3$

b. $I = 5$

c. $I = 11$

d. $I = 15$ Correct Choice

e. $I = 33$

Solution: Green's Theorem says: $\int_{\partial R} P\,dx + Q\,dy = \iint_R (\partial_x Q - \partial_y P)\,dA$.

We identify: $P = 3y + 2xy^3$ and $Q = 8x + 3x^2y^2$. So $\partial_x Q - \partial_y P = (8 + 6xy^2) - (3 + 6xy^2) = 5$.

Consequently: $I = \iint_R 5\,dA = 5Area = 5 \cdot 3 = 15$

2. Compute $\int_{(1,1,1)}^{(2,4,8)} \vec{F} \cdot d\vec{s}$ for $\vec{F} = \langle 2xyz, x^2z, x^2y \rangle$ along the curve $\vec{r}(t) = \langle t, t^2 t^3 \rangle$.

HINT: Find a scalar potential.

a. 128

b. 127 Correct Choice

c. 0

d. -127

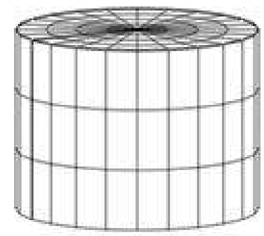
e. -128

Solution: By inspection, a scalar potential is $f = x^2yz$ since $\vec{\nabla} f = \langle 2xyz, x^2z, x^2y \rangle = \vec{F}$.

By the Fundamental Theorem of Calculus for Curves,

$$\int_{(1,1,1)}^{(2,4,8)} \vec{F} \cdot d\vec{s} = \int_{(1,1,1)}^{(2,4,8)} \vec{\nabla} f \cdot d\vec{s} = f(2,4,8) - f(1,1,1) = 2^2 \cdot 4 \cdot 8 - 1^2 \cdot 1 \cdot 1 = 127$$

3. Compute $\iint_{\partial R} \vec{F} \cdot d\vec{S}$ over the complete surface of the cylinder $x^2 + y^2 \le 4$ for $0 \le z \le 3$ with **outward** orientation, for $\vec{F} = \langle xy^4, x^4y, 2x^2y^2z \rangle$.
HINT: Use a theorem.



a. 64π Correct Choice

b. 48π

c. $\dfrac{192}{5}\pi$

d. 24π

e. 16π

Solution: $\vec{\nabla} \cdot \vec{F} = y^4 + x^4 + 2x^2y^2 = (x^2+y^2)^2$ In cylindrical coordinates, $\vec{\nabla} \cdot \vec{F} = r^4$ and $dV = r\,dr\,d\theta\,dz$.
By Gauss' Theorem,

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \vec{\nabla} \cdot F\,dV = \int_0^3 \int_0^{2\pi} \int_0^2 r^4 r\,dr\,d\theta\,dz = 2\pi[z]_0^3 \left[\frac{r^6}{6}\right]_0^2 = 2\pi \cdot 3 \cdot \frac{2^6}{6} = 64\pi$$

4. The two legs of a right triangle are $\vec{a}$ and $\vec{b}$ and the hypotenuse is $\vec{c}$. So $\vec{a} \perp \vec{b}$ and $\vec{c} = \vec{a} + \vec{b}$. Given that $\vec{c} = \langle 12, -12, 12 \rangle$ and the direction of $\vec{a}$ is $\hat{a} = \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{-1}{3} \right\rangle$, find the magnitude $\vec{b}$.

a. $\left|\vec{b}\right| = 48$

b. $\left|\vec{b}\right| = 24\sqrt{2}$

c. $\left|\vec{b}\right| = 24$

d. $\left|\vec{b}\right| = 12\sqrt{2}$ Correct Choice

e. $\left|\vec{b}\right| = 12$

Solution: $\vec{a}$ is the projection of $\vec{c}$ onto $\hat{a}$. Since $\vec{c} \cdot \hat{a} = 8 + 8 - 4 = 12$ and $|\hat{a}| = 1$, we have:

$$\vec{a} = proj_{\hat{a}}\vec{c} = \frac{\vec{c} \cdot \hat{a}}{|\hat{a}|^2}\hat{a} = \frac{12}{1}\left\langle \frac{2}{3}, \frac{-2}{3}, \frac{-1}{3} \right\rangle = \langle 8, -8, -4 \rangle$$

Then $\vec{b} = \vec{c} - \vec{a} = \langle 12, -12, 12 \rangle - \langle 8, -8, -4 \rangle = \langle 4, -4, 16 \rangle$. So $\left|\vec{b}\right| = \sqrt{16 + 16 + 256} = 12\sqrt{2}$.

5. An ant is walking across a frying pan where the temperature is $T = x^3y^2$. If the ant is currently at $P = (2,3)$ and has velocity $\vec{v} = \langle 2,-4\rangle$, what is the rate of change of the temperature as seen by the ant?

a. 408

b. 204

c. 24 Correct Choice

d. 12

e. 6

Solution: $\vec{\nabla}T = \langle 3x^2y^2, 2x^3y\rangle = \langle 108,48\rangle$ $\qquad \dfrac{dT}{dt} = \vec{v}\cdot\vec{\nabla}T = \langle 2,-4\rangle\cdot\langle 108,48\rangle = 216 - 192 = 24$

6. The point $(1,-2)$ is a critical point of the function $f = \frac{16}{y} - \frac{8}{x} - x^2y^2$.

Classify the point $(1,-2)$ using the Second Derivative Test.

a. Local Minimum

b. Local Maximum Correct Choice

c. Inflection Point

d. Saddle Point

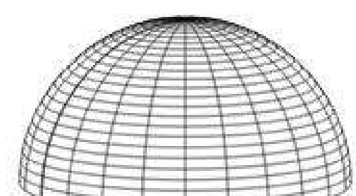
e. Test Fails

Solution: $f_x = \dfrac{8}{x^2} - 2xy^2 \qquad f_x(1,-2) = 8 - 8 = 0 \qquad f_y = -\dfrac{16}{y^2} - 2x^2y \qquad f_y(1,-2) = -4 + 4 = 0$

$f_{xx} = -\dfrac{16}{x^3} - 2y^2 = -24 < 0 \qquad f_{yy} = \dfrac{32}{y^3} - 2x^2 = -6 < 0 \qquad f_{xy} = -4xy = 8$

$D = f_{xx}f_{yy} - {f_{xy}}^2 = 144 - 64 = 80 > 0$ Local Maximum

7. Find the mass of the solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$ if the density is $\delta = z$.



a. 2π Correct Choice

b. 4π

c. 8π

d. 16π

e. 32π

Solution: In spherical coordinates, $\delta = \rho\cos\varphi$ and $dV = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$. So

$$M = \iiint \delta\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho\cos\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2}\left[\frac{\rho^4}{4}\right]_0^2 = 2\pi\frac{1}{2}4 = 4\pi$$

8. Find the center of mass of the solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$ if the density is $\delta = z$.

a. $\left(0, 0, \frac{64\pi}{15}\right)$

b. $\left(0, 0, \frac{32\pi}{15}\right)$

c. $\left(0, 0, \frac{15}{32\pi}\right)$

d. $\left(0, 0, \frac{15}{16}\right)$

e. $\left(0, 0, \frac{16}{15}\right)$ Correct Choice

Solution: By symmetry, $\bar{x} = \bar{y} = 0$. To find $\bar{z}$ we compute:

$$M_z = \iiint z\,\delta\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2\cos^2\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\left[\frac{-\cos^3\varphi}{3}\right]_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_0^2 = 2\pi\frac{1}{3}\frac{32}{5} = \frac{64\pi}{15}$$

$$\bar{z} = \frac{M_z}{M} = \frac{64\pi}{15}\frac{1}{4\pi} = \frac{16}{15}$$

9. Find the equation of the line perpendicular to the hyperboloid $xyz = 6$ at the point $(3, 2, 1)$.

a. $2x + 3y + 6z = 18$

b. $3x + 2y + z = 18$

c. $(x,y,z) = (3 + 2t, 2 - 3t, 1 + 6t)$

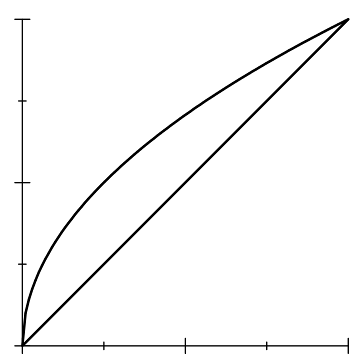
d. $(x,y,z) = (2 + 3t, 3 + 2t, 6 + t)$

e. $(x,y,z) = (3 + 2t, 2 + 3t, 1 + 6t)$ Correct Choice

Solution: $P = (3, 2, 1)$ $\quad F = xyz \quad \vec{\nabla}F = (yz, xz, xy) \quad \vec{N} = \vec{\nabla}F\Big|_{(3,2,1)} = (2, 3, 6)$

$$X = P + t\vec{N} = (3, 2, 1) + t(2, 3, 6) = (3 + 2t, 2 + 3t, 1 + 6t)$$

10. Find the volume under the surface $z = 2x^2y$ above the region bounded by $y = x$ and $y = 2\sqrt{x}$.
The base is shown at the right.



a. $\frac{64}{7}$

b. $\frac{320}{7}$

c. $\frac{256}{5}$ Correct Choice

d. $\frac{64}{5}$

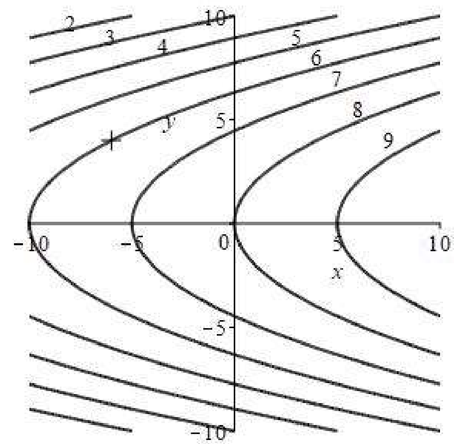
e. $\frac{320}{3}$

Solution: The curves intersect when $x = 2\sqrt{x}$ or $x^2 = 4x$ or $x = 0, 4$

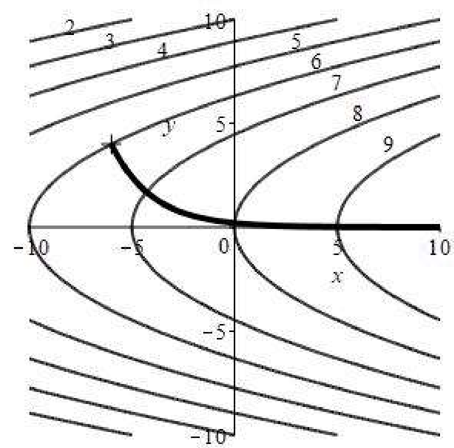
$$V = \int_0^4\int_x^{2\sqrt{x}} 2x^2y\,dy\,dx = \int_0^4 [x^2y^2]_{y=x}^{2\sqrt{x}}\,dx = \int_0^4 (4x^3 - x^4)\,dx = \left[x^4 - \frac{x^5}{5}\right]_{x=0}^4 = \frac{4^4}{5} = \frac{256}{5}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

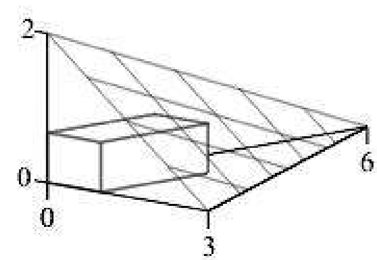
11. (5 points) At the right is the contour plot of a function $f(x,y)$. The contours are labeled by the function values. If you start at the cross at $(-6,4)$ and move so that your velocity is always in the direction of $\vec{\nabla} f$, the gradient of f, roughly sketch your path on the plot.



Solution: You are to draw a curve which starts at the cross, comes down and curves to the right, always perpendicular to each contour it crosses.



12. (20 points) Find the volume of the largest rectangular box in the first quadrant with three faces in the coordinate planes and one vertex on the plane $2x+y+3z=6$.
You do NOT need to check it is a maximum rather than a minimum.



Solution: Minimize $V=xyz$ subject to the constraint $g=2x+y+3z=6$.
Lagrange Multipliers: $\vec{\nabla} V=\langle yz,xz,xy\rangle$ $\vec{\nabla} g=\langle 2,1,3\rangle$ Lagrange equations: $\vec{\nabla} V=\lambda\vec{\nabla} g$

$$yz=2\lambda \quad xz=\lambda \quad xy=3\lambda \qquad \text{or} \qquad \lambda=xz \quad yz=2xz \quad xy=3xz$$

$$\text{or} \quad y=2x \quad y=3z \qquad \text{or} \qquad x=\frac{1}{2}y \quad z=\frac{1}{3}y$$

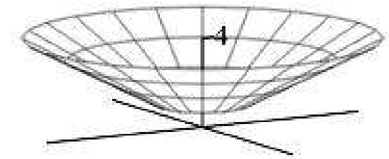
Plug into the constraint, solve for y and substitute back:

$$2x+y+3z=y+y+y=3y=6 \qquad \text{or} \qquad y=2 \quad x=1 \quad z=\frac{2}{3}$$

So the volume is $V=(1)(2)\left(\frac{2}{3}\right)=\frac{4}{3}$

13. (25 points) Verify Stokes' Theorem $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$

for the vector field $\vec{F} = \langle 2yz, -2xz, z^2 \rangle$ and the **surface** which is the piece of the cone C given by $z = \frac{1}{2}\sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$ oriented down and out.



Notice that the boundary of C is two circles.

Be sure to check orientations. Use the following steps:

a. Parametrize the cone by $\vec{R}(r,\theta) = \left\langle r\cos\theta, r\sin\theta, \frac{1}{2}r \right\rangle$.

What is the range or r?

$\underline{\quad 2 \quad} \le r \le \underline{\quad 8 \quad}$

b. Compute the tangent vectors:

$$\begin{array}{ccccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = \dfrac{\partial \vec{R}}{\partial r} & = \Big\langle & \cos\theta, & \sin\theta, & \dfrac{1}{2} \quad \Big\rangle \\ \vec{e}_\theta = \dfrac{\partial \vec{R}}{\partial \theta} & = \langle & -r\sin\theta, & r\cos\theta, & 0 \quad \rangle \end{array}$$

c. Compute the normal vector and check, explain and fix the orientation:

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}\left(0 - \frac{1}{2}r\cos\theta\right) - \hat{\jmath}\left(- -\frac{1}{2}r\sin\theta\right) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = \left\langle -\frac{r}{2}\cos\theta, -\frac{r}{2}\sin\theta, r \right\rangle$$

This is in (x and y are negative in the first quadrant) and up (z is positive). We want down and out.

So we reverse it: $\vec{N} = \left\langle \frac{r}{2}\cos\theta, \frac{r}{2}\sin\theta, -r \right\rangle$

d. Compute the curl of $\vec{F}$ and evaluate it on the surface:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2yz & -2xz & z^2 \end{vmatrix} = \hat{\imath}(0 - -2x) - \hat{\jmath}(-2y) + \hat{k}(-2z - 2z) = \langle 2x, 2y, -4z \rangle$$

$$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = \langle 2r\cos\theta, 2r\sin\theta, -2r \rangle$$

e. Compute the dot product:

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = r^2\cos^2\theta + r^2\sin^2\theta + 2r^2 = 3r^2$$

f. Compute the flux integral:

$$\iint_C \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_2^8 3r^2\, dr\, d\theta = 2\pi[r^3]_2^8 = 2\pi(512 - 8) = 1008\pi$$

Recall $\vec{F} = \langle 2yz, -2xz, z^2 \rangle$

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g. Parametrize the upper circle U:

$\vec{r}(\theta) = \langle 8\cos\theta, 8\sin\theta, 4 \rangle$

h. Compute the tangent vector and check, explain and fix the orientation:

$\vec{v}(\theta) = \langle -8\sin\theta, 8\cos\theta, 0 \rangle$ oriented counterclockwise need clockwise

Reverse it: $\vec{v}(\theta) = \langle 8\sin\theta, -8\cos\theta, 0 \rangle$

i. Evaluate the vector field on the curve:

$\vec{F}\Big|_{\vec{r}(\theta)} = \langle 64\sin\theta, -64\cos\theta, 16 \rangle$

j. Compute the dot product:

$\vec{F} \cdot \vec{v} = 512\sin^2\theta + 512\cos^2\theta = 512$

k. Compute the integal around U:

$$\oint_U \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 512\, d\theta = 1024\pi$$

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l. Parametrize the lower circle L:

$\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 1 \rangle$

m. Compute the tangent vector and check, explain and fix the orientation:

$\vec{v}(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$ oriented counterclockwise correct

n. Evaluate the vector field on the curve:

$\vec{F}\Big|_{\vec{r}(\theta)} = \langle 4\sin\theta, -4\cos\theta, 1 \rangle$

o. Compute the dot product:

$\vec{F} \cdot \vec{v} = -8\sin^2\theta - 8\cos^2\theta = -8$

p. Compute the integal around L:

$$\oint_L \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -8\, d\theta = -16\pi$$

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q. Combine the line integrals:

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \oint_U \vec{F} \cdot d\vec{s} + \oint_L \vec{F} \cdot d\vec{s} = 1024\pi - 16\pi = 1008\pi$$ They agree!