Name____________________________

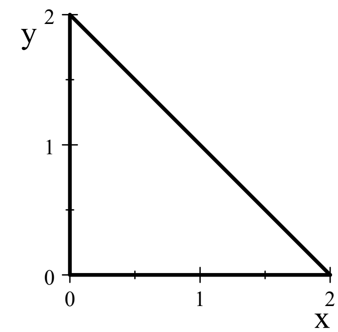
MATH 251 Final Version H Fall 2018

Sections 200/202 Solutions P. Yasskin

1-12	/48	15	/12
13	/ 5	16	/25
14	/20	Total	/110

Multiple Choice: (4 points each. No part credit.)

1. Compute $I = \int_{\partial R} (2y + 3x^2y^2)\,dx + (5x + 2x^3y)\,dy$
over the complete boundary of the triangle shown at the right traversed counterclockwise.
HINT: Use a theorem.



a. $I = 3$

b. $I = 6$ Correct Choice

c. $I = 7$

d. $I = 12$

e. $I = 14$

Solution: Green's Theorem says: $\int_{\partial R} P\,dx + Q\,dy = \iint_R (\partial_x Q - \partial_y P)\,dA$.
We identify: $P = 2y + 3x^2y^2$ and $Q = 5x + 2x^3y$. So $\partial_x Q - \partial_y P = (5 + 6x^2y) - (2 + 6x^2y) = 3$.
Consequently: $I = \iint_R 3\,dA = 3Area = 3 \cdot \frac{1}{2} \cdot b \cdot h = 3 \cdot \frac{1}{2} \cdot 2 \cdot 2 = 6$

2. Compute $\int_{(1,1,1)}^{(8,4,2)} \vec{F} \cdot d\vec{s}$ for $\vec{F} = \langle y^2z^2, 2xyz^2, 2xy^2z \rangle$ along the curve $\vec{r}(t) = \langle t^3, t^2, t \rangle$.
HINT: Find a scalar potential.

a. -512

b. -511

c. 0

d. 511 Correct Choice

e. 512

Solution: By inspection, a scalar potential is $f = xy^2z^2$ since $\vec{\nabla} f = \langle y^2z^2, 2xyz^2, 2xy^2z \rangle = \vec{F}$.
By the Fundamental Theorem of Calculus for Curves,

$$\int_{(1,1,1)}^{(8,4,2)} \vec{F} \cdot d\vec{s} = \int_{(1,1,1)}^{(8,4,2)} \vec{\nabla} f \cdot d\vec{s} = f(8,4,2) - f(1,1,1) = 8 \cdot 4^2 \cdot 2^2 - 1 \cdot 1^2 \cdot 1^2 = 511$$

3. Compute $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the quartic surface $z = x^4 + 2x^2y^2 + y^4$ with $z \le 16$ oriented **down** and **out**, for $\vec{F} = \langle -yz, xz, z^2 \rangle$.
HINT: Use a theorem.

a. 128π Correct Choice

b. 256π

c. 512π

d. 1024π

e. 2048π

Solution: Stokes' Theorem says $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$. In cylindrical coordinates, the surface can be written as $z = (x^2 + y^2)^2 = r^4$. So its boundary is $z = r^4 = 16$ or $r = 2$. This is a circle which may be parametrized as $\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 16 \rangle$. Then $\vec{v} = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$. On the circle, $\vec{F} = \langle -32\sin\theta, 32\cos\theta, 16^2 \rangle$. So $\vec{F} \cdot \vec{v} = 64\sin^2\theta + 64\cos^2\theta + 0 = 64$. Then

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 64\, d\theta = 128\pi$$

4. The two legs of a right triangle are $\vec{a}$ and $\vec{b}$ and the hypotenuse is $\vec{c}$. So $\vec{a} \perp \vec{b}$ and $\vec{c} = \vec{a} + \vec{b}$. Given that $\vec{c} = \langle 9, 9, -9 \rangle$ and the direction of $\vec{a}$ is $\hat{a} = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \right\rangle$, find the magnitude $\vec{b}$.

a. $\left|\vec{b}\right| = 36$

b. $\left|\vec{b}\right| = 18\sqrt{2}$

c. $\left|\vec{b}\right| = 18$

d. $\left|\vec{b}\right| = 9\sqrt{2}$ Correct Choice

e. $\left|\vec{b}\right| = 9$

Solution: $\vec{a}$ is the projection of $\vec{c}$ onto $\hat{a}$. Since $\vec{c} \cdot \hat{a} = 6 - 3 + 6 = 9$ and $|\hat{a}| = 1$, we have:

$$\vec{a} = proj_{\hat{a}}\vec{c} = \frac{\vec{c} \cdot \hat{a}}{|\hat{a}|^2}\hat{a} = \frac{9}{1}\left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \right\rangle = \langle 6, -3, -6 \rangle$$

Then $\vec{b} = \vec{c} - \vec{a} = \langle 9, 9, -9 \rangle - \langle 6, -3, -6 \rangle = \langle 3, 12, -3 \rangle$. So $\left|\vec{b}\right| = \sqrt{9 + 144 + 9} = 9\sqrt{2}$.

5. An ant is walking across a frying pan where the temperature is $T = \frac{1}{12}x^3y^2$. If the ant is currently at $P = (2,3)$, in what unit vector direction should the ant walk to reduce the temperature as fast as possible?

a. $\left\langle \frac{9}{13}, \frac{4}{13} \right\rangle$

b. $\left\langle \frac{-9}{5}, \frac{-4}{5} \right\rangle$

c. $\left\langle \frac{9}{5}, \frac{4}{5} \right\rangle$

d. $\left\langle \frac{9}{\sqrt{97}}, \frac{4}{\sqrt{97}} \right\rangle$

e. $\left\langle \frac{-9}{\sqrt{97}}, \frac{-4}{\sqrt{97}} \right\rangle$ Correct Choice

Solution: $\vec{\nabla}T = \left\langle \frac{1}{4}x^2y^2, \frac{1}{6}x^3y \right\rangle = \langle 9,4 \rangle \qquad \vec{u} = -\vec{\nabla}T = \langle -9,-4 \rangle$

$|\vec{u}| = \sqrt{9^2+4^2} = \sqrt{97} \qquad \hat{u} = \frac{\vec{u}}{|\vec{u}|} = \left\langle \frac{-9}{\sqrt{97}}, \frac{-4}{\sqrt{97}} \right\rangle$

6. The point $(1,2)$ is a critical point of the function $f(x,y) = 16x^4 + y^4 - 32xy$.
Classify the point $(1,2)$ using the Second Derivative Test.

a. Local Mininum Correct Choice

b. Local Maximum

c. Saddle Point

d. Inflection Point

e. Test Fails

Solution: $f_x = 64x^3 - 32y \qquad f_x(1,2) = 64-64 = 0 \qquad f_y = 4y^3 - 32x \qquad f_y(1,2) = 32-32 = 0$

$f_{xx} = 192x^2 \qquad f_{xx}(1,2) = 192 > 0 \qquad f_{yy} = 12y^2 \qquad f_{yy}(1,2) = 48 \qquad f_{xy} = -32 \qquad f_{xy}(1,2) = -32$

$D = f_{xx}f_{yy} - f_{xy}{}^2 \qquad D(1,2) = 192 \cdot 48 - 32^2 = 8192 > 0$ Local Minimum

7. Find the mass of the piece of the solid paraboloid $z = x^2 + y^2$ for $2 \le z \le 4$ if the density is $\delta = z$.

a. 64π

b. 60π

c. $\frac{112}{3}\pi$

d. $\frac{56}{3}\pi$ Correct Choice

e. 20π

Solution: In cylindrical coordinates, $dV = r\,dr\,d\theta\,dz$ and the paraboloid is $z = r^2$. Since z goes between constant limits, we put the z integral outside and write the cone as $r = \sqrt{z}$. So

$$M = \iiint \delta\,dV = \int_0^{2\pi}\int_2^4\int_0^{\sqrt{z}} z\,r\,dr\,dz\,d\theta = 2\pi\int_2^4 z\left[\frac{r^2}{2}\right]_0^{\sqrt{z}} dz = \pi\int_2^4 z^2\,dz = \pi\frac{z^3}{3}\Big|_2^4 = \pi\frac{64-8}{3} = \frac{56\pi}{3}$$

8. Find the center of mass of the piece of the solid paraboloid $z = x^2 + y^2$ for $2 \le z \le 4$ if the density is $\delta = z$.

a. $\frac{14}{75}$

b. $\frac{14}{15}$

c. $\frac{45}{14}$ Correct Choice

d. $\frac{14}{45}$

e. $\frac{15}{14}$

Solution: By symmetry, $\bar{x} = \bar{y} = 0$. To find $\bar{z}$ we compute:

$$M_z = \iiint z\,\delta\,dV = \int_0^{2\pi}\int_2^4\int_0^{\sqrt{z}} z^2\,r\,dr\,dz\,d\theta = 2\pi\int_2^4 z^2\left[\frac{r^2}{2}\right]_0^{\sqrt{z}} dz = \pi\int_2^4 z^3\,dz = \pi\frac{z^4}{4}\Big|_2^4 = \pi\frac{256-16}{4} = 60\pi$$

$$\bar{z} = \frac{M_z}{M} = 60\pi\frac{3}{56\pi} = \frac{45}{14}$$

9. Find the equation of the plane tangent to the hyperboloid $xyz = 6$ at the point $(3,2,1)$.

a. $(x,y,z) = (3+2t, 2+3t, 1+6t)$

b. $(x,y,z) = (2+3t, 3+2t, 6+t)$

c. $3x + 2y + z = 14$

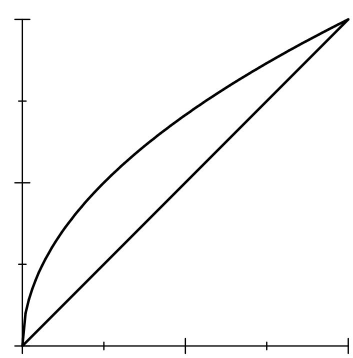
d. $3x + 2y + z = 18$

e. $2x + 3y + 6z = 18$ Correct Choice

Solution: $P = (3,2,1)$ $\quad F = xyz$ $\quad \vec{\nabla}F = (yz, xz, xy)$ $\quad \vec{N} = \vec{\nabla}F\Big|_{(3,2,1)} = (2,3,6)$

$N \cdot X = N \cdot P \qquad 2x + 3y + 6z = 2(3) + 3(2) + 6(1) = 18$

10. Find the volume under the surface $z = 2xy$ above the region bounded by $y = x$ and $y = 2\sqrt{x}$.
The base is shown at the right.



a. $\frac{128}{3}$

b. $\frac{128}{5}$

c. $\frac{64}{3}$ Correct Choice

d. $\frac{64}{5}$

e. $\frac{64}{7}$

Solution: The curves intersect when $x = 2\sqrt{x}$ or $x^2 = 4x$ or $x = 0, 4$

$$V = \int_0^4\int_x^{2\sqrt{x}} 2xy\,dy\,dx = \int_0^4 [xy^2]_{y=x}^{2\sqrt{x}}\,dx = \int_0^4 (4x^2 - x^3)\,dx = \left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_{x=0}^4 = 4^4\left(\frac{4-3}{12}\right) = \frac{64}{3}$$

11. Compute $\begin{vmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 & 0 \end{vmatrix}$

a. $2!$

b. $3!$

c. $4!$

d. $5!$ Correct Choice

e. $6!$

Solution: Expand on the 1^{st}, then 2^{nd}, then 3^{rd}, then 3^{rd} rows. Each time the checkerboard gives a minus:

$$\begin{vmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 & 0 \end{vmatrix} = -2\begin{vmatrix} 3 & 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix} = 6\begin{vmatrix} 3 & 0 & 5 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 \end{vmatrix} = -24\begin{vmatrix} 3 & 0 & 5 \\ 1 & 0 & 2 \\ 0 & 5 & 0 \end{vmatrix} = 120\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 120$$

12. In $\mathbb{R}^2$, compute $\oint_C \vec{F} \cdot d\vec{n}$, the flux of $\vec{F} = \langle x^3, y^3 \rangle$ outward thru the circle $x^2 + y^2 = 4$.

HINT: Use a Theorem.

a. 12π

b. 16π

c. 24π Correct Choice

d. 36π

e. 48π

Solution: Let D be the disk whose boundary is the circle C. By the 2D Gauss' Theorem,

$$\oint_C \vec{F} \cdot d\vec{n} = \iint_D \vec{\nabla} \cdot \vec{F}\, dA = \iint_D (3x^2 + 3y^2)\, dx\, dy = \int_0^{2\pi} \int_0^2 3r^3\, dr\, d\theta = 6\pi\left[\frac{r^4}{4}\right]_0^2 = 24\pi$$

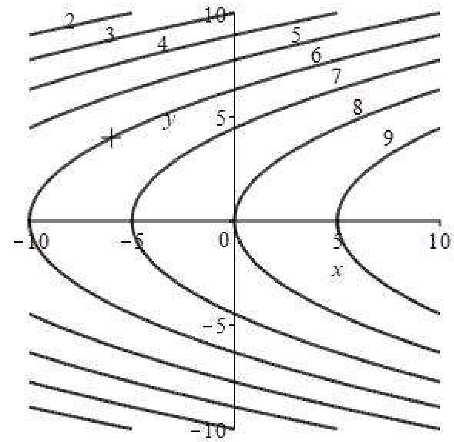
Alternatively, by Green's Theorem, (with $P = -y^3$ and $Q = x^3$)

$$\oint_C \vec{F} \cdot d\vec{n} = \oint_C x^3\, dy - y^3\, dx = \oint_C P\, dx + Q\, dy = \iint_D (\partial_x Q - \partial_y P)\, dA = \iint_D (3x^2 + 3y^2)\, dx\, dy$$

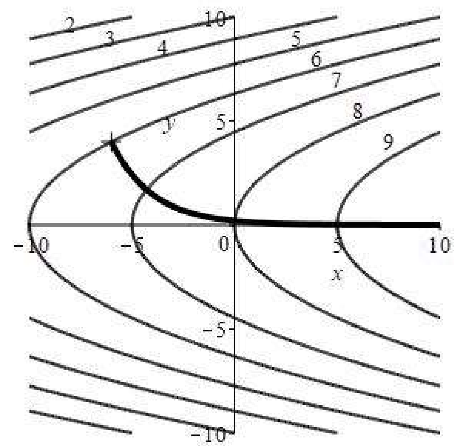
which gives the same thing.

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (5 points) At the right is the contour plot of a function $f(x,y)$. The contours are labeled by the function values. If you start at the cross at $(-6,4)$ and move so that your velocity is always in the direction of $\vec{\nabla} f$, the gradient of f, roughly sketch your path on the plot.

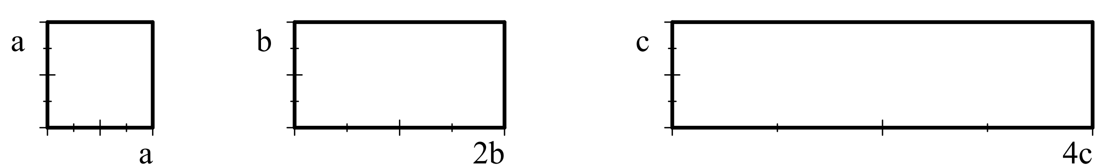


Solution: You are to draw a curve which starts at the cross, comes down and curves to the right, always perpendicular to each contour it crosses.



14. (20 points) A 118 cm wire is cut into 3 pieces. As shown in the plots, one piece is bent into a square of side a. Another piece is bent into a rectangle with sides b and $2b$. The third piece is bent into a rectangle with sides c and $4c$. Note: a, b and c may not be the same length, even if they look that way in the plots. Find a, b and c which minimize the total area enclosed in the three shapes. Note: the constraint is the sum of the perimeters.

You do NOT need to check it is a minimum rather than a maximum.



Solution: Minimize $A = a^2 + 2b^2 + 4c^2$ subject to the constraint $L = 4a + 6b + 10c = 118$.

Lagrange Multipliers: $\vec{\nabla} A = \langle 2a, 4b, 8c \rangle$ $\vec{\nabla} L = \langle 4, 6, 10 \rangle$ Lagrange equations: $\vec{\nabla} A = \lambda \vec{\nabla} L$

$$2a = 4\lambda \qquad 4b = 6\lambda \qquad 8c = 10\lambda \qquad \text{or} \qquad a = 2\lambda \qquad b = \frac{3}{2}\lambda \qquad c = \frac{5}{4}\lambda$$

Plug into the constraint:

$$4a + 6b + 10c = 8\lambda + 9\lambda + \frac{25}{2}\lambda = 118$$

Solve for λ and substitute back:

$$16\lambda + 18\lambda + 25\lambda = 236 \qquad \lambda = \frac{236}{59} = 4 \qquad a = 8 \qquad b = 6 \qquad c = 5$$

15. (12 points) Determine whether or not each of these limits exists. If it exists, find its value.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^6+3y^3}$

SOLUTION: Straight line approaches: $y = mx$

$$\lim_{\substack{y=mx\\x\to 0}} \frac{3x^2y^2}{x^6+3y^3} = \lim_{x\to 0}\frac{3x^2m^2x^2}{x^6+3m^3x^3} = \lim_{x\to 0}\frac{3m^2x}{x^3+3m^3} = \frac{0}{3m^3} = 0$$

Quadratic approaches: $y = mx^2$

$$\lim_{\substack{y=mx^2\\x\to 0}} \frac{3x^2y^2}{x^6+3y^3} = \lim_{x\to 0}\frac{3x^2m^2x^4}{x^6+3m^3x^6} = \lim_{x\to 0}\frac{3m^2}{1+3m^3} \neq 0 \qquad \text{if } m \neq 0.$$

Limit does not exist because these are different.

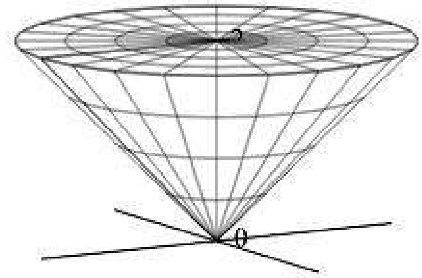
b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$

SOLUTION: Switch to polar: $x = r\cos\theta \quad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{\substack{r\to 0\\ \theta \text{ arbitrary}}} \frac{r\cos\theta\, r^2\sin^2\theta}{r^2} = \lim_{\substack{r\to 0\\ \theta \text{ arbitrary}}} r\cos\theta\sin^2\theta = 0$$

because $r \to 0$ while $\cos\theta\sin^2\theta$ is bounded: $-1 \le \cos\theta\sin^2\theta \le 1$.

16. (25 points) Verify Gauss' Theorem $\iiint\limits_V \vec{\nabla} \bullet \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \bullet d\vec{S}$



for the vector field $\vec{F} = \langle -xz^2, -yz^2, z^3 \rangle$ and the solid above the cone $z = \sqrt{x^2+y^2}$ below the plane $z = 2$.

Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence and give the volume element in the appropriate coordinate system:

$\vec{\nabla} \bullet \vec{F} = -z^2 - z^2 + 3z^2 = z^2$ $\qquad dV = r\,dr\,d\theta\,dz$

b. Compute the left hand side: Here are 2 ways:

$$\iiint\limits_V \vec{\nabla} \bullet \vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_0^z z^2\, r\,dr\,dz\,d\theta = 2\pi\int_0^2 \left[z^2 \frac{r^2}{2}\right]_{r=0}^z dz = \pi\int_0^2 z^4\,dz = \pi\left[\frac{z^5}{5}\right]_0^2 = \frac{32\pi}{5}$$

$$\iiint\limits_V \vec{\nabla} \bullet \vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_r^2 z^2\, r\,dz\,dr\,d\theta = 2\pi\int_0^2 \left[\frac{z^3}{3} r\right]_{z=r}^2 dr = \frac{2\pi}{3}\int_0^2 8r - r^4\,dr$$

$$= \frac{2\pi}{3}\left[4r^2 - \frac{r^5}{5}\right]_0^2 = \frac{2\pi}{3}2^4\left(1 - \frac{2}{5}\right) = \frac{32\pi}{5}$$

Second the Right Hand Side:

The boundary surface consists of the cone C and a disk D with appropriate orientations.

c. Parametrize the disk D:

$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2\rangle$

d. Compute the tangent vectors:

$\vec{e}_r = \langle \cos\theta, \quad \sin\theta, \quad 0\rangle$

$\vec{e}_\theta = \langle -r\sin\theta, \quad r\cos\theta, \quad 0\rangle$

e. Compute the normal vector:

$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = \langle 0, 0, r\rangle$ Need up. The orientation is correct.

f. Evaluate $\vec{F} = \langle -xz^2, -yz^2, z^3\rangle$ on the disk:

$\vec{F}\Big|_{\vec{R}(r,\theta)} = \langle -4r\cos\theta, -4r\sin\theta, 8\rangle$

g. Compute the dot product:

$\vec{F} \bullet \vec{N} = 8r$

h. Compute the flux through D:

$$\iint\limits_D \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^2 8r\,dr\,d\theta = 2\pi[4r^2]_0^2 = 32\pi$$

Parametrize the cone C as $\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$

i. Compute the tangent vectors:

$$\vec{e}_r = \left\langle \cos\theta, \quad \sin\theta, \quad 1 \right\rangle$$

$$\vec{e}_\theta = \langle -r\sin\theta, \quad r\cos\theta, \quad 0\rangle$$

j. Compute the normal vector:

$$\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(- -r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

This is in and up. We need out and down.

Reverse: $\vec{N} = \langle r\cos\theta, r\sin\theta, -r\rangle$

k. Evaluate $\vec{F} == \langle -xz^2, -yz^2, z^3\rangle$ on the cone:

$$\vec{F}\Big|_{\vec{R}(\theta,\varphi)} = \langle -r^3\cos\theta, -r^3\sin\theta, r^3\rangle$$

l. Compute the dot product:

$$\vec{F}\bullet\vec{N} = -r^4\cos^2\theta - r^4\sin^2\theta - r^4 = -2r^4$$

m. Compute the flux through C:

$$\iint_C \vec{F}\bullet d\vec{S} = \int_0^{2\pi}\int_0^2 -2r^4\,dr\,d\theta = 2\pi\left[\frac{-2r^5}{5}\right]_0^2 = \frac{-128\pi}{5}$$

n. Compute the **TOTAL** right hand side:

$$\iint_{\partial V} \vec{F}\bullet d\vec{S} = \iint_D \vec{F}\bullet d\vec{S} + \iint_C \vec{F}\bullet d\vec{S} = 32\pi - \frac{128\pi}{5} = \frac{32\pi}{5}$$ which agrees with (c).