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MATH 253 Exam 1 Spring 2003

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (5 points each) Work Out: (points indicated)

1. Find the distance between the points $P = (2,6,-3)$ and $Q = (6,3,9)$.

a. $\sqrt{151}$
b. 13 correctchoice
c. 19
d. 151
e. 169

$\overrightarrow{PQ} = Q - P = \langle 4,-3,12\rangle$ dist $= \left|\overrightarrow{PQ}\right| = \sqrt{4^2+3^2+12^2} = \sqrt{169} = 13$

2. A triangle has vertices $A = (1,2,3)$, $B = (1,0,1)$ and $C = (2,2,2)$. Find the angle at A.

a. $0°$
b. $30°$
c. $45°$
d. $60°$ correctchoice
e. $120°$

$\overrightarrow{AB} = B - A = \langle 0,-2,-2\rangle$ $\overrightarrow{AC} = C - A = \langle 1,0,-1\rangle$ $\left|\overrightarrow{AB}\right| = \sqrt{8}$ $\left|\overrightarrow{AC}\right| = \sqrt{2}$ $\overrightarrow{AB}\cdot\overrightarrow{AC} = 2$

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{2}{\sqrt{8}\sqrt{2}} = \frac{1}{2} \qquad \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60°$$

3. A triangle has vertices $A = (1,2,3)$, $B = (1,0,1)$ and $C = (2,2,2)$. Find the area.

a. $\sqrt{6}$
b. $\sqrt{2}$
c. $\sqrt{3}$ correctchoice
d. $2\sqrt{2}$
e. $2\sqrt{3}$

$\overrightarrow{AB} = B - A = \langle 0,-2,-2\rangle$ $\overrightarrow{AC} = C - A = \langle 1,0,-1\rangle$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -2 & -2 \\ 1 & 0 & -1 \end{vmatrix} = \hat{\imath}(2) - \hat{\jmath}(2) + \hat{k}(2) = (2,-2,2)$$

Area $= \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\sqrt{4+4+4} = \frac{1}{2}\sqrt{12} = \sqrt{3}$

4. Find x so that $\langle 2,1,3\rangle \cdot \langle 4,2,x\rangle = 1$.

- **a.** -3 correctchoice
- **b.** -2
- **c.** 0
- **d.** 2
- **e.** 3

$\langle 2,1,3\rangle \cdot \langle 4,2,x\rangle = 10 + 3x = 1 \quad \Rightarrow \quad x = -3$

5. $\vec{a}$ is 4 units long and points along the positive x-axis.
$\vec{b}$ is 3 units long and points along the positive z-axis. Then $\vec{a} \times \vec{b}$ is

- **a.** 7 units long and points along the negative x-axis.
- **b.** 5 units long and points along the positive y-axis.
- **c.** 5 units long and points along the negative y-axis.
- **d.** 12 units long and points along the positive y-axis.
- **e.** 12 units long and points along the negative y-axis. correctchoice

$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \sin 90° = 12$

By Right Hand Rule $\vec{a} \times \vec{b}$ points along $\hat{\imath} \times \hat{k} = -\hat{\jmath}$, the negative y-axis.

6. A certain plane may be parametrized as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$.

Find its non-parametric equation.

- **a.** $x + y + z = 6$
- **b.** $x - y - z = 0$
- **c.** $x - y + z = 2$ correctchoice
- **d.** $-x - y + z = -4$
- **e.** $x + y - z = -2$

$$\vec{N} = (1,0,-1) \times (0,2,2) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 2 & 2 \end{vmatrix} = \hat{\imath}(2) - \hat{\jmath}(2) + \hat{k}(2) = (2,-2,2)$$

$X = (3,2,1) \qquad P = (3,2,1)$

$\vec{N} \cdot X = \vec{N} \cdot P \qquad 2x - 2y + 2z = 6 - 4 + 2 = 4 \qquad x - y + z = 2$

7. You swing a ball on a string, so that it moves in the circle $\vec{r}(t) = (3\cos t, 0, 3\sin t)$ where t measures time. What is the tangential acceleration?

a. 0 correctchoice
b. 3
c. 6
d. 9
e. 12

$\vec{v}(t) = (-3\sin t, 0, 3\cos t)$ $|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t} = 3$ $a_T = \dfrac{d|\vec{v}|}{dt} = 0$

8. If $f(x,y,z) = x^4y^3z^2$ then $\left.\dfrac{\partial^4 f}{\partial^2 x\,\partial y\,\partial z}\right|_{(x,y,z)=(1,2,3)} =$

a. 1
b. 6
c. 72
d. 96
e. 864 correctchoice

$\dfrac{\partial^4 f}{\partial^2 x\,\partial y\,\partial z} = 4 \cdot 3 \cdot x^2 \cdot 3 \cdot y^2 \cdot 2 \cdot z = 72x^2y^2z$ $\left.\dfrac{\partial^4 f}{\partial^2 x\,\partial y\,\partial z}\right|_{(x,y,z)=(1,2,3)} = 72(1)^2(2)^2(3) = 864$

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9. $(13$ points$)$ Find the arc length of the curve $\vec{r}(t) = \left\langle \frac{3}{2}t^2, 2t^2, \frac{5}{3}t^3 \right\rangle$ for $0 \le t \le \sqrt{3}$.
HINT: Factor the quantity inside the square root.

$\vec{v}(t) = \langle 3t, 4t, 5t^2 \rangle$ $|\vec{v}| = \sqrt{9t^2 + 16t^2 + 25t^4} = 5\sqrt{t^2 + t^4} = 5t\sqrt{1 + t^2}$

$L = \int_0^{\sqrt{3}} 5t\sqrt{1 + t^2}\, dt$ $u = 1 + t^2$ $du = 2t\,dt$

$L = \dfrac{5}{2}\int_{t=0}^{\sqrt{3}} \sqrt{u}\, du = \dfrac{5}{2}\dfrac{2u^{3/2}}{3} = \dfrac{5}{3}(1 + t^2)^{3/2}\Big|_0^{\sqrt{3}} = \dfrac{5}{3}(4^{3/2} - 1) = \dfrac{35}{3}$

10. $(12$ points$)$ The equation $xz^2 + yz^3 = 5$ implicitly defines z as a function of x and y. Compute $\frac{\partial z}{\partial y}$ at the point $(x,y,z) = (3,2,1)$.

By implicit differentiation, product rule and chain rule, $x2z\frac{\partial z}{\partial y} + z^3 + y3z^2\frac{\partial z}{\partial y} = 0$

Solve for $\frac{\partial z}{\partial y}$: $(2xz + 3yz^2)\frac{\partial z}{\partial y} = -z^3$ $\qquad \frac{\partial z}{\partial y} = \frac{-z^3}{2xz + 3yz^2}$

At $(x,y,z) = (3,2,1)$: $\frac{\partial z}{\partial y} = \frac{-z^3}{2xz + 3yz^2} = \frac{-1^3}{2 \cdot 3 \cdot 1 + 3 \cdot 2 \cdot 1^2} = \frac{-1}{12}$

11. $(20$ points$)$ Consider the hyperbolic paraboloid $z = f(x,y) = y^2 - x^2$. Find the tangent plane at $(x,y) = (1,2)$. Then identify the z-intercept of the tangent plane.

$f(1,2) = 3 \qquad f_x = -2x \qquad f_x(1,2) = -2 \qquad f_y = 2y \qquad f_y(1,2) = 4$

Tangent Plane:
$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 3 - 2(x-1) + 4(y-2) = -2x + 4y - 3$

z-intercept: $x = y = 0 \qquad z = -3$

12. $(15$ points$)$ The equation $2xz^4 + yz^3 = 7$ implicitly defines $z = f(x,y)$ near $(x,y,z) = (2,3,1)$. Using implicit differentiation, it can be shown (DON'T DO IT.) that

$$\frac{\partial f}{\partial x}(2,3) = -.08 \quad \text{and} \quad \frac{\partial f}{\partial y}(2,3) = -.04$$

Using the linear approximation, estimate $f(2.1,3.2)$.

$f(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) = 1 - .08(x-2) - .04(y-3)$

$f(2.1,3.2) = 1 - .08(2.1-2) - .04(3.2-3) = 1 - .08(.1) - .04(.2) = 1 - .008 - .008 = .984$

13. $(5$ points Extra Credit$)$ Drop a perpendicular from the point $Q = (4,6)$ to the line $(x,y) = (1,2) + t(2,1)$. Find the foot of the perpendicular, i.e. the point where the perpendicular line intersects the original line.

A point on the line is $P = (1,2)$. The direction of the line is $\vec{v} = (2,1)$.

The point we want is $X = P + proj_{\vec{v}}\overrightarrow{PQ}$. We compute:

$\overrightarrow{PQ} = Q - P = (3,4) \qquad \overrightarrow{PQ} \cdot \vec{v} = (3,4) \cdot (2,1) = 10 \qquad |\vec{v}| = \sqrt{5}$

$proj_{\vec{v}}\overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} = \frac{10}{5}(2,1) = (4,2)$

$X = P + proj_{\vec{v}}\overrightarrow{PQ} = (1,2) + (4,2) = (5,4)$