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	11	/15			

MATH 253 Exam 2 Spring 2003

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (5 points each) Work Out: (15 points each)

1. The density of the air in a room is given by $\rho = x^2 z + y^2 z$. At $t = 3$, a fly's position is $\vec{r} = (3,3,4)$ and its velocity is $\vec{v} = (2,3,-1)$. Find $\dfrac{d\rho}{dt}$ as seen by the fly at $t = 3$.

a. -39
b. -11
c. 11
d. 39
e. 102 correctchoice

$\vec{\nabla}\rho = (2xz, 2yz, x^2 + y^2)$ $\vec{\nabla}\rho(3,3,4) = (24,24,18)$ $\dfrac{d\rho}{dt} = \vec{v} \cdot \vec{\nabla}\rho = (2,3,-1) \cdot (24,24,18) = 102$

2. The altitude on a mountain is $z = 27 - 3x^2 - 5y^2 + xy$. A hiker is currently on the mountain at the point $(x,y) = (2,1)$. In what direction should the hiker walk to come down as quickly as possible?

a. $(8,-11)$
b. $(8,11)$
c. $(11,8)$ correctchoice
d. $(11,-8)$
e. $(-11,-8)$

$\vec{\nabla}z = (-6x + y, -10y + x)$ $\vec{\nabla}z(2,1) = (-11,-8)$
The hiker must walk in the direction $-\vec{\nabla}z(2,1) = (11,8)$.

3. (Extra Credit) The altitude on a mountain is $z = 27 - 3x^2 - 5y^2 + xy$. A hiker is currently on the mountain at the point $(x,y) = (2,1)$. In what direction should the hiker walk to stay at exactly the same altitude?

a. $(8,-11)$ correctchoice
b. $(8,11)$
c. $(11,8)$
d. $(11,-8)$
e. $(-11,-8)$

From problem 3, $\vec{\nabla}z(2,1) = (-11,-8)$. To stay at the same altitude, the hiker must walk along a contour line (level curve), i.e. perpendicular to $\vec{\nabla}z$. The vector $(8,-11)$ is perpendicular to $\vec{\nabla}z$.

4. Suppose $f(x,y) = x^2y^3$ where $x = x(u,v)$ and $y = y(u,v)$. Compute $\left.\frac{\partial f}{\partial v}\right|_{(u,v)=(3,4)}$ if

$$x(3,4) = 2 \qquad \left.\frac{\partial x}{\partial u}\right|_{(u,v)=(3,4)} = 4 \qquad \left.\frac{\partial x}{\partial v}\right|_{(u,v)=(3,4)} = 6$$

$$y(3,4) = 1 \qquad \left.\frac{\partial y}{\partial u}\right|_{(u,v)=(3,4)} = 3 \qquad \left.\frac{\partial y}{\partial v}\right|_{(u,v)=(3,4)} = 5$$

- **a.** 17
- **b.** 39
- **c.** 72
- **d.** 84 correctchoice
- **e.** 4464

$$\left.\frac{\partial f}{\partial v}\right|_{(3,4)} = \left.\frac{\partial f}{\partial x}\right|_{(x(3,4),y(3,4))}\left.\frac{\partial x}{\partial v}\right|_{(3,4)} + \left.\frac{\partial f}{\partial y}\right|_{(x(3,4),y(3,4))}\left.\frac{\partial y}{\partial v}\right|_{(3,4)} = 2xy^3|_{(2,1)}6 + 3x^2y^2|_{(2,1)}5 = 24 + 60 = 84$$

5. Find the equation of the plane tangent to the surface $xz^2 + yz^3 = 5$ at the point $(x,y,z) = (2,3,1)$.

- **a.** $x + y + 13z = 18$ correctchoice
- **b.** $2x + 3y + z = 18$
- **c.** $2x + 9y + 5z = 36$
- **d.** $2x + 3y + z = 36$
- **e.** $2x + 3y + z = 14$

$$F = xz^2 + yz^3 \qquad \vec{\nabla}F = (z^2, z^3, 2xz + 3yz^2) \qquad \vec{N} = \left.\vec{\nabla}F\right|_{(2,3,1)} = (1,1,13)$$

$$\vec{N} \cdot X = \vec{N} \cdot P \qquad x + y + 13z = 2 + 3 + 13 = 18$$

6. Which of the following is NOT a critical point of the function $g = xy^3 + x^3y - 9xy$?

- **a.** $(0,0)$
- **b.** $(-3,0)$
- **c.** $(0,3)$
- **d.** $(0,-3)$
- **e.** $(-3,3)$ correctchoice

$$\vec{\nabla}g = (-9y + y^3 + 3x^2y, -9x + x^3 + 3xy^2)$$

By trial and error, $\vec{\nabla}g = 0$ at all of the answers except $(-3,3)$.

7. The function $f = xy^3 + x^3y - 4xy$ has a critical point at the point $(x, y) = (1, -1)$. Use the Second Derivative Test to classify this critical point.

a. Local Minimum
b. Local Maximum correctchoice
c. Saddle Point
d. Test Fails

$\vec{\nabla} f = (y^3 + 3x^2y - 4y, 3xy^2 + x^3 - 4x)$

Hess(f) $= \begin{pmatrix} 6xy & 3x^2 + 3y^2 - 4 \\ 3x^2 + 3y^2 - 4 & 6xy \end{pmatrix} = \begin{pmatrix} -6 & 2 \\ 2 & -6 \end{pmatrix}$

$f_{xx} = -6 < 0 \qquad D = 36 - 4 = 32 > 0 \qquad \Rightarrow$ Local Maximum

8. Compute $\int_0^2 \int_y^{y^2} 2xy\,dx\,dy$.

a. $\frac{56}{15}$
b. $\frac{10}{3}$
c. $\frac{20}{3}$ correctchoice
d. $\frac{32}{3}$
e. $\frac{56}{3}$

$$\int_0^2 \int_y^{y^2} 2xy\,dx\,dy = \int_0^2 \left[x^2y\right]_{x=y}^{x=y^2} dy = \int_0^2 (y^5 - y^3)\,dy = \left[\frac{y^6}{6} - \frac{y^4}{4}\right]_0^2 = \frac{32}{3} - 4 = \frac{20}{3}$$

9. Compute $\int_0^2 \int_0^3 \int_{x^2+y^2}^{2x^2+2y^2} 1\,dz\,dy\,dx$.

a. 13
b. 26 correctchoice
c. 31
d. 32
e. 62

$$\int_0^2 \int_0^3 \int_{x^2+y^2}^{2x^2+2y^2} 1\,dz\,dy\,dx = \int_0^2 \int_0^3 \left[z\right]_{x^2+y^2}^{2x^2+2y^2} dy\,dx = \int_0^2 \int_0^3 ([2x^2 + 2y^2] - [x^2 + y^2])\,dy\,dx$$
$$= \int_0^2 \int_0^3 (x^2 + y^2)\,dy\,dx = \int_0^2 \left[x^2y + \frac{y^3}{3}\right]_0^3 dy\,dx = \int_0^2 (3x^2 + 9)\,dy\,dx = \left[x^3 + 9x\right]_0^2 = 8 + 18 = 26$$

10. Find the length L, width W and height H of the rectangular box with maximum volume such that $L + 2W + 3H = 36$. You must solve by **eliminating a variable**. (Do not solve by Lagrange multipliers.)

$$L + 2W + 3H = 36 \quad \Rightarrow \quad L = 36 - 2W - 3H$$

Maximize $V = LWH = (36 - 2W - 3H)WH = 36WH - 2W^2H - 3H^2W$

$$\frac{\partial V}{\partial W} = 36H - 4WH - 3H^2 = H(36 - 4W - 3H) = 0$$

$$\frac{\partial V}{\partial H} = 36W - 2W^2 - 6HW = W(36 - 2W - 6H) = 0$$

Since the volume is non-zero, $W \neq 0$ and $H \neq 0$. So
$$4W + 3H = 36 \quad \cdots(\text{eq } 1)$$
$$2W + 6H = 36 \quad \cdots(\text{eq } 2)$$

$2 \times (\text{eq } 2) - (\text{eq } 1)$: $\quad 9H = 36 \quad H = 4$

$2 \times (\text{eq } 1) - (\text{eq } 2)$: $\quad 6W = 36 \quad W = 6$

$L = 36 - 2W - 3H = 36 - 12 - 12 = 12$

11. A styrofoam cup in the shape of a cylinder without a lid is to hold 64π cm^3 of coffee. Find the radius r and height h of the cup which uses the least styrofoam. Ignore the thickness of the styrofoam. You must solve by **Lagrange multipliers**. (Do not solve by eliminating a variable.)

Maximize: $\quad A = 2\pi rh + \pi r^2 \qquad \vec{\nabla} A = (2\pi h + 2\pi r, 2\pi r)$

Constraint: $\quad V = \pi r^2 h = 64\pi \qquad \vec{\nabla} V = (2\pi rh, \pi r^2)$

Lagrange equations: $\quad \vec{\nabla} A = \lambda \vec{\nabla} V \qquad \begin{pmatrix} 2\pi h + 2\pi r = \lambda 2\pi rh \\ 2\pi r = \lambda \pi r^2 \end{pmatrix}$

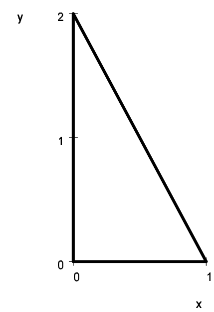
Since the volume is non-zero, $r \neq 0$ and $h \neq 0$. So $\begin{pmatrix} h + r = \lambda rh & \cdots(\text{eq } 1) \\ 2 = \lambda r & \cdots(\text{eq } 2) \end{pmatrix}$

Use $(\text{eq } 2)$ to eliminate λr from $(\text{eq } 1)$: $\quad h + r = 2h \quad$ So $r = h$.

Use the constraint: $\quad \pi r^3 = 64\pi \qquad r^3 = 64 \qquad r = 4 \qquad h = 4$

12. Find the volume below surface $z = 2xy$ above the triangle with vertices $(0,0)$, $(1,0)$ and $(0,2)$ in the xy-plane.

Plot the triangle. The diagonal edge is $y = 2 - 2x$.



$$V = \int_0^1 \int_0^{2-2x} 2xy\, dy\, dx = \int_0^1 \left[xy^2 \right]_{y=0}^{2-2x} dx$$

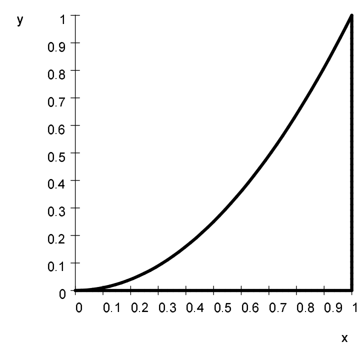
$$= \int_0^1 x(2-2x)^2\, dx = \int_0^1 x(4 - 8x + 4x^2)\, dx$$

$$= \int_0^1 (4x - 8x^2 + 4x^3)\, dx = \left[2x^2 - \frac{8}{3}x^3 + x^4 \right]_0^1$$

$$= 2 - \frac{8}{3} + 1 = \frac{1}{3}$$

13. Compute $I = \int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy$ by interchanging the order of integration.

Plot the region. The diagonal edge is $x = \sqrt{y}$ or $y = x^2$.



$$I = \int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy = \int_0^1 \int_0^{x^2} e^{x^3}\, dy\, dx$$

$$= \int_0^1 \left[e^{x^3} y \right]_0^{x^2} dx = \int_0^1 e^{x^3} x^2\, dx$$

Let $u = x^3$. Then $du = 3x^2\, dx$ and $\frac{1}{3} du = x^2\, dx$

$$I = \frac{1}{3} \int e^u\, du = \frac{1}{3} e^u = \left[\frac{1}{3} e^{x^3} \right]_0^1 = \frac{e}{3} - \frac{1}{3}$$