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MATH 253 Exam 3 Spring 2003

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (6 points each) Work Out: (13 points each) Extra Credit: (6 points)

1. If $\vec{G} = (x\sin z, y\sin z, xy\cos z)$, then $\vec{\nabla} \cdot \vec{G} =$

a. $(2 - xy)\sin z$ correctchoice
b. $-xy\sin z$
c. $2x\cos z - 2y\cos z$
d. $(\sin z, -\sin z, -xy\sin z)$
e. $(\sin z, \sin z, -xy\sin z)$

$$\vec{\nabla} \cdot \vec{G} = \partial_x(x\sin z) + \partial_y(y\sin z) + \partial_z(xy\cos z) = \sin z + \sin z - xy\sin z = (2 - xy)\sin z$$

2. If $\vec{G} = (x\sin z, y\sin z, xy\cos z)$, then $\vec{\nabla} \times \vec{G} =$

a. $(\sin z, -\sin z, -xy\sin z)$
b. $\big((x - y)\cos z, (y - x)\cos z, 0\big)$
c. $\big((x - y)\cos z, (x - y)\cos z, 0\big)$ correctchoice
d. $2x\cos z - 2y\cos z$
e. 0

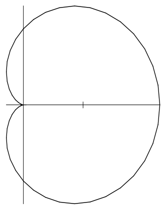
$$\vec{\nabla} \times \vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x\sin z & y\sin z & xy\cos z \end{vmatrix} = \hat{\imath}(x\cos z - y\cos z) - \hat{\jmath}(y\cos z - x\cos z) + \hat{k}(0 - 0)$$
$$= \big((x - y)\cos z, (x - y)\cos z, 0\big)$$

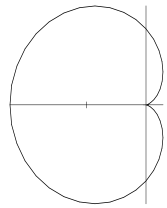
3. If $\vec{F} = \vec{\nabla} f$ and $\vec{\nabla} \times \vec{F} = (12x^2, 12y^2, 12z^2)$ then which of the following could be f?

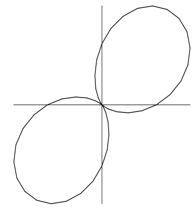
a. $x^4 + y^4 + z^4$
b. $x^4y^4z^4$
c. $x^3y^3 + y^3z^3 + z^3x^3$
d. $x^2y^2 + y^2z^2 + z^2x^2$
e. None of the above correctchoice

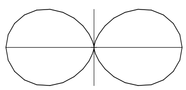
For any f, $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla} f = 0$. So (a)-(d) are impossible.

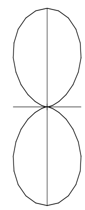
4. Which of the following is the graph of $r = 1 + \cos 2\theta$

a. 

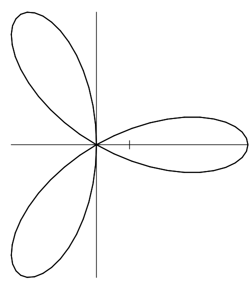
b. 

c. 

d.  ← correctchoice

e. 

5. Compute the area inside one leaf of the 3-leaf rose $r = \cos 3\theta$.



a. $\frac{\pi}{12}$ correctchoice

b. $\frac{\pi}{6}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{3}$

e. $\frac{\pi}{2}$

$r = 0$ when $\cos 3\theta = 0$ or $3\theta = \pm\frac{\pi}{2}$ or $\theta = \pm\frac{\pi}{6}$

$$A = \iint 1\,dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6} \frac{r^2}{2}\Big|_0^{\cos 3\theta} d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6}\cos^2 3\theta\,d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6}\frac{1+\cos(6\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin(6\theta)}{6}\right]_{-\pi/6}^{\pi/6} = \frac{1}{4}\left[\frac{\pi}{6} - -\frac{\pi}{6}\right] = \frac{\pi}{12}$$

6. Find the total mass of the hemisphere $0 \le z \le \sqrt{9 - x^2 - y^2}$ if the volume density is $\delta = x^2 + y^2 + z^2$.

a. $\dfrac{243\pi}{5}$

b. $\dfrac{486\pi}{5}$ correctchoice

c. 9π

d. $9\pi^2$

e. 18π

In spherical coordinates, the density is $\delta = \rho^2$.

$$M = \iiint \delta\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \rho^2\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = [2\pi][-\cos\varphi]_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_0^3 = [2\pi][1]\left[\frac{243}{5}\right] = \frac{486\pi}{5}$$

7. Find the arc length of the curve $r(t) = \left(2t, t^2, \frac{1}{3}t^3\right)$ between $t = 0$ and $t = 2$.

a. 2

b. $\dfrac{7}{3}$

c. $\dfrac{20}{3}$ correctchoice

d. 36

e. 60

$$\vec{v} = (2, 2t, t^2) \qquad |\vec{v}| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2 + t^2)^2} = 2 + t^2$$

$$L = \int_0^2 1\, ds = \int_0^2 |\vec{v}|\, dt = \int_0^2 (2 + t^2)\, dt = \left[2t + \frac{t^3}{3}\right]_0^2 = 4 + \frac{8}{3} = \frac{20}{3}$$

8. Compute $\displaystyle\int_{(0,0,0)}^{(4,4,8/3)} (xy + 3z)\, ds$ along the curve $r(t) = \left(2t, t^2, \frac{1}{3}t^3\right)$.

a. 2

b. $\dfrac{7}{3}$

c. $\dfrac{20}{3}$

d. 36

e. 56 correctchoice

$$\vec{v} = (2, 2t, t^2) \qquad |\vec{v}| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2 + t^2)^2} = 2 + t^2$$

$$xy + 3z = 2t^3 + t^3 = 3t^3$$

$$\int_{(0,0,0)}^{(4,4,8/3)} (xy + 3z)\, ds = \int_0^2 3t^3(2 + t^2)\, dt = \int_0^2 (6t^3 + 3t^5)\, dt = \left[\frac{6t^4}{4} + \frac{3t^6}{6}\right]_0^2 = 24 + 32 = 56$$

9. Compute $\int_{(0,0,0)}^{(4,4,8/3)} \vec{F} \cdot d\vec{s}$ along the curve $r(t) = \left(2t, t^2, \frac{1}{3}t^3\right)$ where $\vec{F} = (3z, 2y, x)$.

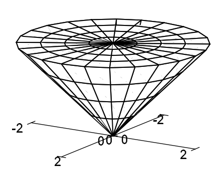
- **a.** 4
- **b.** 8
- **c.** 16
- **d.** 32 correctchoice
- **e.** 64

$\vec{v} = (2, 2t, t^2) \qquad \vec{F} = (3z, 2y, x) = (t^3, 2t^2, 2t)$

$$\int_{(0,0,0)}^{(4,4,8/3)} \vec{F} \cdot d\vec{s} = \int_0^2 \vec{F} \cdot \vec{v}\, dt = \int_0^2 (2t^3 + 4t^3 + 2t^3)\, dt = \int_0^2 8t^3\, dt = 2t^4 \Big|_0^2 = 32$$

10. The solid cone $\sqrt{x^2 + y^2} \le z \le 2$ is shown at the right.

Find the mass and center of mass of the cone if its volume density is given by $\delta = x^2 + y^2$.



In cyindrical coordinates, the cone is $r \le z \le 2$ and the density is $\delta = r^2$.

So the mass is (Don't forget the Jacobian r.)

$$M = \iiint_C \delta\, dV = \int_0^{2\pi} \int_0^2 \int_r^2 r^2\, r\, dz\, dr\, d\theta = 2\pi \int_0^2 r^3 z \Big|_r^2 dr = 2\pi \int_0^2 [2r^3 - r^4]\, dr = 2\pi \left[\frac{r^4}{2} - \frac{r^5}{5} \right]_0^2$$

$$= 2\pi \left[8 - \frac{32}{5} \right] = 16\pi \left[1 - \frac{4}{5} \right] = \frac{16\pi}{5}$$

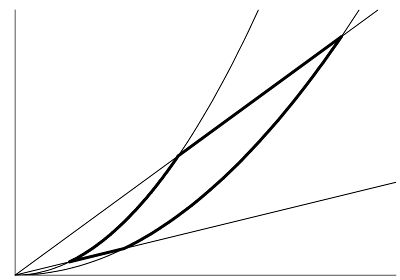
By symmetry, $\bar{x} = \bar{y} = 0$.

$$z\text{-mom} = \iiint_C z\, \delta\, dV = \int_0^{2\pi} \int_0^2 \int_r^2 z\, r^2\, r\, dz\, dr\, d\theta = 2\pi \int_0^2 r^3 \frac{z^2}{2} \Big|_r^2 dr = 2\pi \int_0^2 \left[2r^3 - \frac{r^5}{2} \right] dr$$

$$= 2\pi \left[\frac{r^4}{2} - \frac{r^6}{12} \right]_0^2 = 2\pi \left[8 - \frac{16}{3} \right] = 16\pi \left[1 - \frac{2}{3} \right] = \frac{16\pi}{3}$$

$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{16\pi}{3} \frac{5}{16\pi} = \frac{5}{3}$$

11. Consider the "diamond shaped" region D bounded by the lines $y = x$ and $y = 3x$ and the parabolas $y = x^2$ and $y = 2x^2$. Compute $\iint_D \frac{y}{x}\, dA$ over the diamond.



Here are some steps to follow:

- Let $u = \frac{y}{x}$ and $v = \frac{y}{x^2}$. Solve for x and y.

$\frac{u}{v} = \frac{y}{x}\frac{x^2}{y} = x$ $\qquad$ $\frac{u^2}{v} = \left(\frac{y}{x}\right)^2 \frac{x^2}{y} = y$ $\qquad$ So $x = \frac{u}{v}$ and $y = \frac{u^2}{v}$.

- Find the Jacobian factor.

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ \frac{2u}{v} & -\frac{u^2}{v^2} \end{pmatrix} \right| = \left| -\frac{u^2}{v^3} - -\frac{2u^2}{v^3} \right| = \frac{u^2}{v^3}$$

- Express the integrand in terms of u and v.

$$\frac{y}{x} = \frac{u^2/v}{u/v} = u$$

- Express the boundary curves in terms of u and v.

$y = x$ and $y = 3x$ become $u = 1$ and $u = 3$.

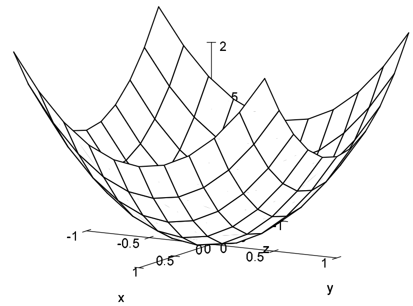
$y = x^2$ and $y = 2x^2$ become $v = 1$ and $v = 2$.

- Compute $\iint_D \frac{y}{x}\, dA$.

$$\iint_D \frac{y}{x}\, dA = \int_1^2 \int_1^3 u \frac{u^2}{v^3}\, du\, dv = \int_1^2 v^{-3}\, dv \int_1^3 u^3\, du = \left[\frac{v^{-2}}{-2}\right]_1^2 \left[\frac{u^4}{4}\right]_1^3$$

$$= \left[-\frac{1}{8} + \frac{1}{2}\right]\left[\frac{81}{4} - \frac{1}{4}\right] = \frac{3}{8}20 = \frac{15}{2}$$

12. Find the total mass of the parametric surface



$$\vec{R}(p,q) = (p, q, p^2 + q^2)$$

for $-1 \le p \le 1$ and $-1 \le q \le 1$

if the surface density is

$$\delta = \sqrt{4z+1}\,.$$

HINT: Find $\vec{e}_p$, $\vec{e}_q$, $\vec{N}$ and δ on the surface.

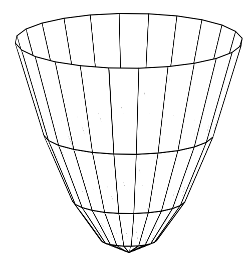
$$\begin{array}{lccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_p = & (1 & 0 & 2p) \\ \vec{e}_q = & (0 & 1 & 2q) \end{array} \qquad \vec{N} = \vec{e}_p \times \vec{e}_q = (-2p, -2q, 1) \qquad \left|\vec{N}\right| = \sqrt{4p^2 + 4q^2 + 1}$$

$$\delta = \sqrt{4(p^2+q^2)+1}$$

$$M = \iint \delta\, dS = \int\!\!\int \delta \left|\vec{N}\right| dp\, dq = \int_{-1}^{1}\int_{-1}^{1} (4p^2 + 4q^2 + 1)\, dp\, dq = \int_{-1}^{1} \left[\frac{4p^3}{3} + 4q^2 p + p \right]_{p=-1}^{1} dq$$

$$= \int_{-1}^{1} \left[\frac{8}{3} + 8q^2 + 2 \right]_{-1}^{1} dq = \left[\frac{8q}{3} + \frac{8q^3}{3} + 2q \right]_{q=-1}^{1} = \left[\frac{16}{3} + \frac{16}{3} + 4 \right] = \frac{44}{3}$$

13. Compute $\displaystyle\iint_P \vec{F} \cdot d\vec{S}$ for $\vec{F} = (xz, yz, z^2)$ over



the surface of the paraboloid P given by

$$z = x^2 + y^2 \le 16$$

with normal pointing down and out.

The paraboloid may be parametrized by

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2).$$

HINT: Find $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$ and $\vec{F}$ on the surface.

$$\begin{array}{lccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\ \cos\theta & \sin\theta & 2r) \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 0\ ) \end{array} \qquad \vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$\vec{N}$ is up and in. Reverse it: $\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$

On the surface, $\vec{F} = (xz, yz, z^2) = (r^3\cos\theta, r^3\sin\theta, r^4)$

$$\iint_P \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^4 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^4 (2r^5\cos^2\theta + 2r^5\sin^2\theta - r^5)\, dr\, d\theta = \int_0^{2\pi}\int_0^4 r^5\, dr\, d\theta$$

$$= 2\pi \frac{r^6}{6}\bigg|_0^4 = \frac{4^6\pi}{3} = \frac{4096\pi}{3}$$