Name____________________ ID__________ Section______

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MATH 253 Final Exam Spring 2003

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (5 points each) Work Out: (points indicated)

1. Find the angle between the line $\vec{r}(t) = (3+2t,-2,4-2t)$ and the normal to the plane $y-z=4$.

a. $0°$
b. $30°$
c. $45°$
d. $60°$ correctchoice
e. $90°$

$\vec{v} = (2,0,-2)$ $\quad \vec{N} = (0,1,-1)$ $\quad \vec{v}\cdot\vec{N} = 2$ $\quad |\vec{v}| = \sqrt{4+4} = \sqrt{8}$ $\quad \left|\vec{N}\right| = \sqrt{2}$

$$\cos\theta = \frac{\vec{v}\cdot\vec{N}}{|\vec{v}|\left|\vec{N}\right|} = \frac{2}{\sqrt{8}\sqrt{2}} = \frac{1}{2} \qquad \theta = 60°$$

2. Find the volume of the parallepiped with edges $\vec{A} = (1,2,0)$ $\vec{B} = (2,0,4)$ and $\vec{C} = (0,3,-1)$.

a. -16
b. -8
c. 4
d. 8 correctchoice
e. 16

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 4 \\ 0 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} = -12-2(-2) = -8 \qquad V = 8$$

3. Find the equation of the plane tangent to the graph of $f(x,y) = x^3y + xy^2$ at $(2,1)$. Its z-intercept is

a. -38
b. -28 correctchoice
c. 10
d. 35
e. 48

$f_x(x,y) = 3x^2y + y^2 \qquad f_y(x,y) = x^3 + 2xy$

$f(2,1) = 8+2 = 10 \qquad f_x(2,1) = 12+1 = 13 \qquad f_y(2,1) = 8+4 = 12$

$f_{\tan}(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 10 + 13(x-2) + 12(y-1) = 13x + 12y - 28$

4. You are standing at the airport facing North. You look up and see an airplane circling clockwise above the airport. At the moment when the plane is heading due East, in what direction does the plane's binormal point?

a. Up   correctchoice
b. Down
c. North
d. South
e. West

$\vec{T}$ points East. Since the plane is circling clockwise, $\vec{N}$ points North. So $\vec{B} = \vec{T} \times \vec{N}$ points Up.

5. Suppose $z = \frac{x}{y}$, where $x = x(r,s)$ and $y = y(r,s)$ are functions satisfying

$$x(1,2) = 3 \qquad \frac{\partial x}{\partial r}(1,2) = 5 \qquad \frac{\partial x}{\partial s}(1,2) = 7$$
$$y(1,2) = 4 \qquad \frac{\partial y}{\partial r}(1,2) = 6 \qquad \frac{\partial y}{\partial s}(1,2) = 8$$

Find $\frac{\partial z}{\partial r}$ at $(r,s) = (1,2)$.

a. $\frac{-1}{16}$
b. $\frac{1}{8}$ correctchoice
c. $\frac{1}{4}$
d. $\frac{3}{4}$
e. 1

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \qquad \left.\frac{\partial z}{\partial r}\right|_{(1,2)} = \left|\frac{\partial z}{\partial x}\right|_{(3,4)}\left|\frac{\partial x}{\partial r}\right|_{(1,2)} + \left|\frac{\partial z}{\partial y}\right|_{(3,4)}\left|\frac{\partial y}{\partial r}\right|_{(1,2)}$$

$$\frac{\partial z}{\partial x} = \frac{1}{y} \qquad \left|\frac{\partial z}{\partial x}\right|_{(3,4)} = \frac{1}{4} \qquad \frac{\partial z}{\partial y} = \frac{-x}{y^2} \qquad \left|\frac{\partial z}{\partial y}\right|_{(3,4)} = \frac{-3}{16}$$

$$\left.\frac{\partial z}{\partial r}\right|_{(1,2)} = \frac{1}{4} \cdot 5 + \frac{-3}{16} \cdot 6 = \frac{20-18}{16} = \frac{1}{8}$$

6. Compute $\iint\limits_R \frac{1}{x^2+y^2}\,dx\,dy$ over the ring $9 \le x^2 + y^2 \le 16$.

a. $2\pi \ln \frac{16}{9}$
b. $4\pi \ln \frac{16}{9}$
c. $\pi \ln \frac{4}{3}$
d. $2\pi \ln \frac{4}{3}$ correctchoice
e. $4\pi \ln \frac{4}{3}$

$$\iint\limits_R \frac{1}{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_3^4 \frac{1}{r^2}\,r\,dr\,d\theta = 2\pi \ln r\Big|_3^4 = 2\pi \ln \frac{4}{3}$$

7. If $\vec{F} = (xy, yz, zx)$ then $\vec{\nabla} \cdot \vec{F} =$

a. $y + z + x$ correctchoice
b. $y - z + x$
c. $-y + z - x$
d. $(-y, z, -x)$
e. $(-y, -z, -x)$

$$\vec{\nabla} \cdot \vec{F} = \partial_x(xy) + \partial_y(yz) + \partial_z(zx) = y + z + x$$

8. Compute $\oint (3x + 4y)\,dx + (2x - 3y)\,dy$ counterclockwise around the edge of the rectangle $1 \le x \le 4,\ \ 2 \le y \le 6$. HINT: Use Green's Theorem.

a. -36
b. -24 correctchoice
c. 12
d. 24
e. 36

$P = (3x + 4y) \qquad Q = (2x - 3y)$

$$\oint (3x + 4y)\,dx + (2x - 3y)\,dy = \oint P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$= \int_2^6 \int_1^4 (2 - 4)\,dx\,dy = -2(Area) = -2(3 \cdot 4) = -24$$

9. Compute the line integral $\oint y\,dx - x\,dy$ counterclockwise around the semicircle $x^2 + y^2 = 4$ from $(2,0)$ to $(-2,0)$. HINT: Parametrize the circle.

a. -8π
b. -4π correctchoice
c. π
d. 4π
e. 8π

$$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta) \qquad \vec{v} = (-2\sin\theta, 2\cos\theta) \qquad \vec{F} = (y, -x) = (2\sin\theta, -2\cos\theta)$$

$$\oint y\,dx - x\,dy = \oint \vec{F} \cdot \vec{v}\,d\theta = \int_0^\pi -4\sin^2\theta - 4\cos^2\theta\,d\theta = -4\pi$$

10. Compute $\int \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = \left(\frac{1}{x}, \frac{1}{y}\right)$ along the curve $\vec{r}(t) = \left(e^{\cos(t^2)}, e^{\sin(t^2)}\right)$ for $0 \le t \le \sqrt{\pi}$. Note: $\vec{F} = \vec{\nabla}(\ln x + \ln y)$.

a. -2 correctchoice
b. 0
c. $\frac{2}{e}$
d. 1
e. π

$$\vec{r}(0) = (e^{\cos(0)}, e^{\sin(0)}) = (e, 1) \qquad \vec{r}(\sqrt{\pi}) = (e^{\cos(\pi)}, e^{\sin(\pi)}) = \left(\frac{1}{e}, 1\right)$$

$$\int \vec{F} \cdot d\vec{s} = \int_{(e,1)}^{(1/e,1)} \vec{\nabla}(\ln x + \ln y) \cdot d\vec{s} = \Big[\ln x + \ln y\Big]_{(e,1)}^{(1/e,1)} = \left(\ln\frac{1}{e} + \ln 1\right) - (\ln e + \ln 1) = -2$$

11. (10 points) Find 3 positive numbers x, y and z, whose sum is 120 such that $f(x,y,z) = xy^2z^3$ is a maximum.

METHOD 1: Lagrange Multipliers: $x + y + z = 120$

$$f = xy^2z^3 \qquad \vec{\nabla}f = (y^2z^3, 2xyz^3, 3xy^2z^2) \qquad g = x + y + z \qquad \vec{\nabla}g = (1, 1, 1)$$

$$\vec{\nabla}f = \lambda\vec{\nabla}g \quad \Rightarrow \quad y^2z^3 = \lambda, \quad 2xyz^3 = \lambda, \quad 3xy^2z^2 = \lambda \quad \Rightarrow \quad y^2z^3 = 2xyz^3, \quad y^2z^3 = 3xy^2z^2$$

$$\Rightarrow \quad y = 2x, \quad z = 3x \quad \Rightarrow \quad x + y + z = x + 2x + 3x = 120 \quad \Rightarrow \quad 6x = 120 \quad \Rightarrow \quad x = 20$$

$$x = 20, \quad y = 40, \quad z = 60$$

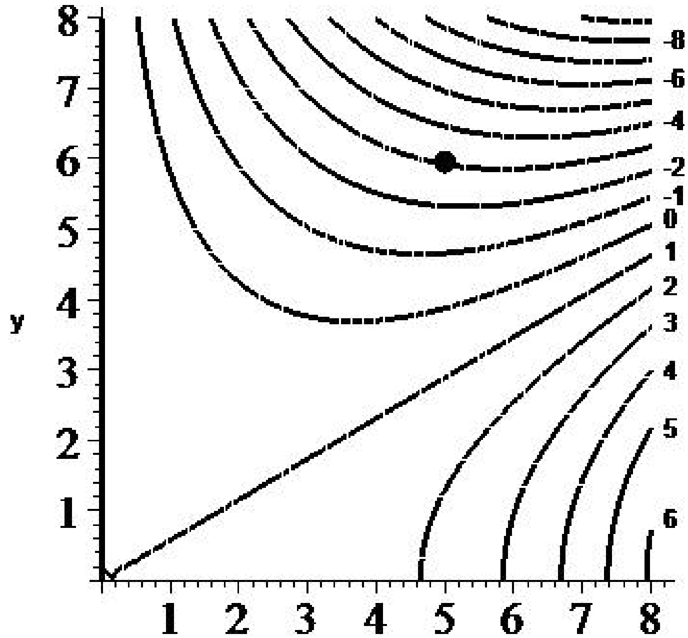
METHOD 2: Eliminate a Variable: $x + y + z = 120$

$$x = 120 - y - z \quad \Rightarrow \quad f = (120 - y - z)y^2z^3 = 120y^2z^3 - y^3z^3 - y^2z^4$$

$$\begin{array}{lcl} f_y = 240yz^3 - 3y^2z^3 - 2yz^4 = 0 & \Rightarrow & 240 - 3y - 2z = 0 \\ f_z = 360y^2z^2 - 3y^3z^2 - 4y^2z^3 = 0 & & 360 - 3y - 4z = 0 \end{array}$$

Subtract: $120 - 2z = 0 \quad \Rightarrow \quad z = 60$
Substitute back: $240 - 3y - 120 = 0 \quad \Rightarrow \quad y = 40$
Substitute back: $x = 120 - y - z = 120 - 40 - 60 = 20$

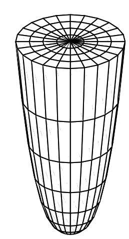
12. (10 points) At the right is the contour plot of a function $f(x,y)$. If you start at the dot at $(5,6)$ and move so that your velocity is always in the direction of $\vec{\nabla} f$, the gradient of f, roughly sketch your path on the plot.



You are to draw a curve which starts at the dot, comes down and curves to the right, always perpendicular to each contour it crosses.

13. (10 points) Find the volume and z-component of the centroid (center of mass) of the solid between the surfaces

$$z = (x^2+y^2)^{3/2} \quad \text{and} \quad z = 8.$$



In cylindrical coordinates: $r^3 \le z \le 8$. So $0 \le r \le 2$.

$$V = \int_0^{2\pi}\int_0^2\int_{r^3}^8 r\,dz\,dr\,d\theta = 2\pi\int_0^2 \Big[rz\Big]_{z=r^3}^8 dr = 2\pi\int_0^2 (8r - r^4)\,dr$$

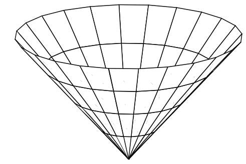
$$= 2\pi\left[4r^2 - \frac{r^5}{5}\right]_0^2 = 2\pi\left(16 - \frac{32}{5}\right) = 32\pi\left(1 - \frac{2}{5}\right) = \frac{96\pi}{5}$$

$$z\text{-mom} = \int_0^{2\pi}\int_0^2\int_{r^3}^8 zr\,dz\,dr\,d\theta = 2\pi\int_0^2\left[r\frac{z^2}{2}\right]_{z=r^3}^8 dr = \pi\int_0^2 (64r - r^7)\,dr$$

$$= \pi\left[32r^2 - \frac{r^8}{8}\right]_0^2 = \pi(128-32) = 96\pi$$

$$\bar{z} = \frac{z\text{-mom}}{V} = 96\pi\frac{5}{96\pi} = 5$$

14. (20 points) Use 2 methods to compute $\iint_C \vec{F} \cdot d\vec{S}$ for $\vec{F} = (5xz, 5yz, z^2)$ over the conical surface C given by $z = \sqrt{x^2 + y^2} \le 3$ with normal pointing down and out.



a. (7 pts) METHOD 1: Compute $\iint_C \vec{F} \cdot d\vec{S}$ directly as a surface integral using the parametrization $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.

HINT: Find $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$ and $\vec{F}$ on the cone.

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r & = & (\ \cos\theta & \sin\theta & 1) \\ \vec{e}_\theta & = & (-r\sin\theta & r\cos\theta & 0) \end{array}$$

$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = (-r\cos\theta, -r\sin\theta, r)$ $\quad \vec{N}$ is up and in.

Reverse it: $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

On the cone, $\vec{F} = (5xz, 5yz, z^2) = (5r^2\cos\theta, 5r^2\sin\theta, r^2)$

$$\iint_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^3 (5r^3\cos^2\theta + 5r^3\sin^2\theta - r^3)\, dr\, d\theta = \int_0^{2\pi}\int_0^3 4r^3\, dr\, d\theta$$

$$= 2\pi r^4\Big|_0^3 = 162\pi$$

b. (13 pts) METHOD 2: Compute $\iint_C \vec{F} \cdot d\vec{S}$ by applying Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$ to the solid cone V whose boundary is $\partial V = C + D$ where C is the conical surface and D is the disk at the top of the cone.

For the volume use cylindrical coordinates. $dV = r\, dr\, d\theta\, dz$ $\quad \vec{\nabla} \cdot \vec{F} = 5z + 5z + 2z = 12z$

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^3\int_r^3 12zr\, dz\, dr\, d\theta = 2\pi\int_0^3 [6z^2]_r^3\, r\, dr = 2\pi\int_0^3 (54 - 6r^2)\, r\, dr$$

$$= 2\pi\int_0^3 (54r - 6r^3)\, dr = 2\pi\left[27r^2 - 6\frac{r^4}{4}\right]_0^3 = 2\pi\left(3^5 - \frac{3^5}{2}\right) = 3^5\pi = 243\pi$$

Parametrize the disk as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 3)$.

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r & = & (\ \cos\theta & \sin\theta & 0) \\ \vec{e}_\theta & = & (-r\sin\theta & r\cos\theta & 0) \end{array}$$

$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = (0, 0, r)$ $\quad \vec{N}$ is up as required.

On the disk, $\vec{F} = (5xz, 5yz, z^2) = (15r\cos\theta, 15r\sin\theta, 9)$

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^3 9r\, dr\, d\theta = 2\pi \frac{9r^2}{2}\Big|_0^3 = 81\pi$$

Apply Gauss' Theorem: $\iint_C \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}\, dV - \iint_D \vec{F} \cdot d\vec{S} = 243\pi - 81\pi = 162\pi$