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MATH 253 Exam 1 Fall 2003

Sections 504-506 Solutions P. Yasskin

Multiple Choice: (6 points each) Work Out: (points indicated)

1. A triangle has vertices $A = (0,3,2)$, $B = (-2,3,0)$ and $C = (-2,0,3)$. Find the angle at vertex B.

a. $\frac{\pi}{6}$

b. $\frac{\pi}{3}$ correctchoice

c. $\frac{\pi}{2}$

d. $\frac{2\pi}{3}$

e. $\frac{5\pi}{6}$

$$\overrightarrow{BA} = A - B = (2,0,2) \qquad \overrightarrow{BC} = C - B = (0,-3,3) \qquad \overrightarrow{BA} \cdot \overrightarrow{BC} = 6$$

$$\left|\overrightarrow{BA}\right| = \sqrt{4+4} = 2\sqrt{2} \qquad \left|\overrightarrow{BC}\right| = \sqrt{9+9} = 3\sqrt{2}$$

$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = \frac{6}{2\sqrt{2}\,3\sqrt{2}} = \frac{1}{2} \qquad \Rightarrow \qquad \theta = 60^\circ = \frac{\pi}{3}$$

2. A triangle has vertices $A = (0,3,2)$, $B = (-2,3,0)$ and $C = (-2,0,3)$. Find the area of the triangle.

a. 15

b. 30

c. $2\sqrt{3}$

d. $3\sqrt{3}$ correctchoice

e. $6\sqrt{3}$

$$\overrightarrow{BA} = A - B = (2,0,2) \qquad \overrightarrow{BC} = C - B = (0,-3,3)$$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 2 \\ 0 & -3 & 3 \end{vmatrix} = \hat{\imath}(6) - \hat{\jmath}(6) + \hat{k}(-6) = (6,-6,-6)$$

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \frac{1}{2}\sqrt{36+36+36} = 3\sqrt{3}$$

3. If $\vec{u}$ points Up (away from the center of the earth) and $\vec{v}$ points NorthEast, then $\vec{u} \times \vec{v}$ points

a. Up

b. Down

c. SouthEast

d. SouthWest

e. NorthWest   correctchoice

Put your fingers Up with the palm facing NorthEast, your thumb points NorthWest.

4. Find the equation of the plane which is perpendicular to the line $(x,y,z) = (2-3t, 3+t, 1-t)$ and passes through the point $(-1,4,3)$.

a. $2x + 3y + z = 13$

b. $2x + 3y + z = -4$

c. $-3x + y - z = 4$ correctchoice

d. $-3x + y - z = -4$

e. $-x + 4y + 3z = 13$

The normal to the plane is the tangent vector to the line: $\vec{N} = \vec{v} = (-3,1,-1)$.
A point on the plane is $P = (-1,4,3)$ So the plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or
$-3x + y - z = 3 + 4 - 3 = 4$

5. Find the arclength of the curve $\vec{r}(t) = \left(\sqrt{2}\,t^2, \sqrt{2}\,t^2, t^3\right)$ between $t = 0$ and $t = 1$.

a. $\frac{61}{27}$ correctchoice

b. $\frac{122}{9}$

c. $\frac{125}{27}$

d. 5

e. $2\ln 2 + \frac{3}{4}\sqrt{5} - 2\ln\left(\sqrt{5} + 1\right)$

$\vec{v} = \left(2\sqrt{2}\,t, 2\sqrt{2}\,t, 3t^2\right) \qquad |\vec{v}| = \sqrt{8t^2 + 8t^2 + 9t^4} = t\sqrt{16 + 9t^2}$

$$L = \int |\vec{v}|\,dt = \int_0^1 t\sqrt{16 + 9t^2}\,dt = \left[\frac{1}{27}(16 + 9t^2)^{3/2}\right]_0^1 = \frac{(25)^{3/2}}{27} - \frac{(16)^{3/2}}{27} = \frac{125 - 64}{27} = \frac{61}{27}$$

6. At time $t = 3$ a fly is at the point $\vec{r}(3) = (2,3,1)$ and has velocity $\vec{v}(3) = (1,2,-1)$. If the fly travels in a straight line, where is the fly at time $t = 5$?

a. $(3,5,0)$

b. $(0,-1,3)$

c. $(1,1,2)$

d. $(4,7,-1)$ correctchoice

e. $(5,8,1)$

The tangent line is $X(s) = r(3) + s\vec{v}(3)$ where $s = 0$ is $t = 3$. Then $t = 5$ is $s = 2$.
So $X(2) = r(3) + 2\vec{v}(3) = (2,3,1) + 2(1,2,-1) = (4,7,-1)$.

7. At time $t = 3$ a fly is at the point $\vec{r}(3) = (2,3,1)$ and has velocity $\vec{v}(3) = (1,2,-1)$. If the density of fly pheromone is given by $P = xy^2 - 2yz^2$, what is the rate of change of the density of fly pheromone as seen by the fly at $t = 3$?

a. 41 correctchoice

b. 36

c. 17

d. 14

e. 1

$\vec{\nabla}P = (y^2, 2xy - 2z^2, -4yz) \qquad \vec{\nabla}P\Big|_{(2,3,1)} = (9,10,-12)$

$\vec{v} \cdot \vec{\nabla}P = (1,2,-1) \cdot (9,10,-12) = 9 + 20 + 12 = 41$

8. At time $t = 3$ a fly is at the point $\vec{r}(3) = (2,3,1)$ and has velocity $\vec{v}(3) = (1,2,-1)$. If the density of fly pheromone is given by $P = xy^2 - 2yz^2$, in what unit vector direction should the fly fly to increase of the density of fly pheromone as fast as possible?

a. $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

b. $\left(\frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

c. $(-9,-10,12)$

d. $\left(\frac{-9}{5\sqrt{13}}, \frac{-2}{\sqrt{13}}, \frac{12}{5\sqrt{13}}\right)$

e. $\left(\frac{9}{5\sqrt{13}}, \frac{2}{\sqrt{13}}, \frac{-12}{5\sqrt{13}}\right)$ correctchoice

$\vec{\nabla}P = (y^2, 2xy - 2z^2, -4yz) \qquad \vec{\nabla}P\Big|_{(2,3,1)} = (9,10,-12) \qquad \left|\vec{\nabla}P\right| = \sqrt{81+100+144} = 5\sqrt{13}$

$$\frac{\vec{\nabla}P}{\left|\vec{\nabla}P\right|} = \left(\frac{9}{5\sqrt{13}}, \frac{2}{\sqrt{13}}, \frac{-12}{5\sqrt{13}}\right)$$

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9. $(10$ points$)$ Consider the set of all points P such that the distance from P to $(3,3,3)$ is twice the distance from P to $(0,0,0)$. This set of points is a sphere. Find its center and radius.

Let $P = (x,y,z)$, $O = (0,0,0)$ and $Q = (3,3,3)$. Then $\left|\overrightarrow{PO}\right| = \sqrt{x^2+y^2+z^2}$ and $\left|\overrightarrow{PQ}\right| = \sqrt{(x-3)^2+(y-3)^2+(z-3)^2}$. The definition of P is $\left|\overrightarrow{PQ}\right| = 2\left|\overrightarrow{PO}\right|$. So

$$\begin{aligned}
\sqrt{(x-3)^2+(y-3)^2+(z-3)^2} &= 2\sqrt{x^2+y^2+z^2} \\
(x-3)^2+(y-3)^2+(z-3)^2 &= 4(x^2+y^2+z^2) \\
x^2+y^2+z^2-6x-6y-6z+27 &= 4x^2+4y^2+4z^2 \\
27 &= 3x^2+3y^2+3z^2+6x+6y+6z \\
x^2+y^2+z^2+2x+2y+2z &= 9 \\
x^2+y^2+z^2+2x+2y+2z+3 &= 12 \\
(x+1)^2+(y+1)^2+(z+1)^2 &= 12
\end{aligned}$$

So the center is $(-1,-1,-1)$ and the radius is $\sqrt{12}$.

10. $(15$ points$)$ Circle whether each limit exists or not. If it exists, find the limit using polar coordinates. If it does not exist find two (or more) curves of approach which give different limits.

a. (10) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$ .... Exists **[Does Not Exist]**

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{x^2+xy+y^2}{x^2+y^2} = \lim_{x\to 0}\frac{x^2+mx^2+m^2x^2}{x^2+m^2x^2} = \lim_{x\to 0}\frac{1+m+m^2}{1+m^2} = \frac{1+m+m^2}{1+m^2}$$

which is different for different m's.

b. (5) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+x^2y+y^2}{x^2+y^2}$ .... **[Exists]** Does Not Exist

$x = r\cos\theta$
$y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+x^2y+y^2}{x^2+y^2} = \lim_{r\to 0}\frac{r^2\cos^2\theta + r^3\cos^2\theta\sin\theta + r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0}\frac{\cos^2\theta + r\cos^2\theta\sin\theta + \sin^2\theta}{\cos^2\theta+\sin^2\theta} = \lim_{r\to 0}(1 + r\cos^2\theta\sin\theta) = 1$$

11. (15 points) Consider the surface $x^2z^3 + y^2z^3 - 74z = -49$.

a. (10) Find the tangent plane at the point $(3,4,1)$.

METHOD 1:

$F = x^2z^3 + y^2z^3 - 74z \qquad \vec{\nabla}F = (2xz^3, 2yz^3, 3x^2z^2 + 3y^2z^2 - 74)$

$\vec{N} = \vec{\nabla}F(3,4,1) = (6,8,27+48-74) = (6,8,1)$

The plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $6x + 8y + z = 6 \cdot 3 + 8 \cdot 4 + 1 = 51$

METHOD 2:

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(3,4,1)$:

$$2xz^3 + x^2 3z^2 \frac{\partial z}{\partial x} + y^2 3z^2 \frac{\partial z}{\partial x} - 74\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \qquad \frac{\partial z}{\partial x} = \frac{-2xz^3}{x^2 3z^2 + y^2 3z^2 - 74} = \frac{-6}{27+48-74} = -6$$

$$x^2 3z^2 \frac{\partial z}{\partial y} + 2yz^3 + y^2 3z^2 \frac{\partial z}{\partial y} - 74\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \qquad \frac{\partial z}{\partial y} = \frac{-2yz^3}{x^2 3z^2 + y^2 3z^2 - 74} = \frac{-8}{27+48-74} = -8$$

The tangent plane to $z = f(x,y)$ is

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = 1 - 6(x-3) - 8(y-4)$

or $z = -6x - 8y + 51$

b. (5) The surface implicitly defines a function $z = f(x,y)$ whose graph (the surface) passes through the point $(3,4,1)$. Use the linear approximation to estimate $f(3.1, 3.9)$.

METHOD 1:

The tangent plane is $z = f_{\tan}(x,y) = 51 - 6x - 8y$.

So $f(3.1,3.9) \approx f_{\tan}(3.1,3.9) = 51 - 6 \cdot 3.1 - 8 \cdot 3.9 = 1.2$

METHOD 2:

The tangent plane is $z = f_{\tan}(x,y) = 1 - 6(x-3) - 8(y-4)$.

So $f(3.1,3.9) \approx f_{\tan}(3.1,3.9) = 1 - 6(3.1-3) - 8(3.9-4) = 1.2$

12. (15 points) Given that $z = x^3 + xy^2$ where $x = x(u,v)$ and $y = y(u,v)$ which satisfy

$$x(3,4) = 1 \qquad \left.\frac{\partial x}{\partial u}\right|_{(3,4)} = 5 \qquad \left.\frac{\partial x}{\partial v}\right|_{(3,4)} = 7$$

$$y(3,4) = 2 \qquad \left.\frac{\partial y}{\partial u}\right|_{(3,4)} = 6 \qquad \left.\frac{\partial y}{\partial v}\right|_{(3,4)} = 8$$

Find $\left.\frac{\partial z}{\partial u}\right|_{(3,4)}$.

$$\frac{\partial z}{\partial x} = 3x^2 + y^2 \qquad \left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 3x^2 + y^2|_{(1,2)} = 3 + 4 = 7 \qquad \frac{\partial z}{\partial y} = 2xy \qquad \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 2xy|_{(1,2)} = 4$$

$$\left.\frac{\partial z}{\partial u}\right|_{(3,4)} = \left.\frac{\partial z}{\partial x}\right|_{(1,2)} \left.\frac{\partial x}{\partial u}\right|_{(3,4)} + \left.\frac{\partial z}{\partial y}\right|_{(1,2)} \left.\frac{\partial y}{\partial u}\right|_{(3,4)} = 7 \cdot 5 + 4 \cdot 6 = 59$$