Name____________________ ID____________ Section______

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MATH 253 Exam 3 Fall 2003

Sections 504-506 Solutions P. Yasskin

Multiple Choice: (5 points each) Work Out: (points indicated)

1. Compute $\iiint x\,dV$ over the solid between the planes $z=0$ and $z=x$, above the triangle with vertices $(0,0)$, $(0,4)$ and $(2,0)$.

a. 4π

b. $\frac{\pi}{2}$

c. $\frac{8}{3}$ correctchoice

d. 4

e. 12

The edges of the triangle are $x=0$, $y=0$ and $y=4-2x$.

$$\iiint x\,dV = \int_0^2\int_0^{4-2x}\int_0^x x\,dz\,dy\,dx = \int_0^2\int_0^{4-2x}\Big[xz\Big]_0^x dy\,dx = \int_0^2\int_0^{4-2x} x^2\,dy\,dx = \int_0^2\Big[x^2y\Big]_0^{4-2x} dx$$

$$= \int_0^2 x^2(4-2x)\,dx = \int_0^2(4x^2-2x^3)\,dx = \left[\frac{4x^3}{3}-\frac{x^4}{2}\right]_0^2 = \frac{32}{3}-8 = \frac{8}{3}$$

2. For the vector field $\vec{F} = (x^2-y^2z, y^2-z^2x, z^2-x^2y)$, compute $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F}$.

a. $4y-4x$

b. $(-2x+2z, -2y+2x, -2z+2y)$

c. $(-2x+2z, 2y-2x, -2z+2y)$

d. $64\pi^2$

e. 0 correctchoice

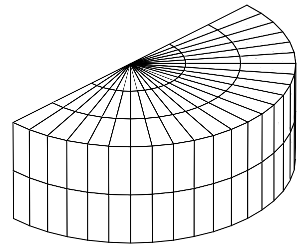
$\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0$ always for any $\vec{F}$. However, in detail:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2-y^2z & y^2-z^2x & z^2-x^2y \end{vmatrix} = \hat{\imath}(-x^2+2zx) - \hat{\jmath}(-2xy+y^2) + \hat{k}(-z^2+2yz)$$

$$= (-x^2+2zx, -y^2+2xy, -z^2+2yz)$$

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = \partial_x(-x^2+2zx)+\partial_y(-y^2+2xy)+\partial_z(-z^2+2yz) = (-2x+2z)+(-2y+2x)+(-2z+2y) = 0$$

3. Compute the mass of the half cylinder $x^2 + y^2 \le 9$ with $y \ge 0$ for $-1 \le z \le 1$ if the density is $\rho = y$.



a. $\frac{3\pi}{2}$

b. $\frac{9\pi}{2}$

c. 9π

d. 18π

e. 36 correctchoice

We use cylindrical coordinates. $\rho = y = r\sin\theta \qquad dV = r\,dr\,d\theta\,dz$

$$M = \iiint \rho\,dV = \int_{-1}^{1}\int_{0}^{\pi}\int_{0}^{3} r\sin\theta\,r\,dr\,d\theta\,dz = \int_{-1}^{1} dz \int_{0}^{\pi} \sin\theta\,d\theta \int_{0}^{3} r^2\,dr$$

$$= \left[z\right]_{-1}^{1}\left[-\cos\theta\right]_{0}^{\pi}\left[\frac{r^3}{3}\right]_{0}^{3} = [2][2][9] = 36$$

4. Compute the y-component of the center of mass of the half cylinder $x^2 + y^2 \le 9$ with $y \ge 0$ for $-1 \le z \le 1$ if the density is $\rho = y$.

a. 9π

b. $\frac{9\pi}{16}$ correctchoice

c. $\frac{81\pi}{4}$

d. $\frac{9}{8}$

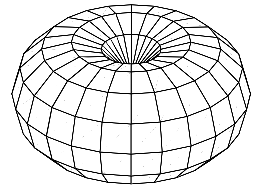
e. $\frac{9}{4}$

$$M_{xz} = \iiint y\rho\,dV = \int_{-1}^{1}\int_{0}^{\pi}\int_{0}^{3} r^2\sin^2\theta\,r\,dr\,d\theta\,dz = \int_{-1}^{1} dz \int_{0}^{\pi} \sin^2\theta\,d\theta \int_{0}^{3} r^3\,dr$$

$$= \left[z\right]_{-1}^{1}\int_{0}^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta\left[\frac{r^4}{4}\right]_{0}^{3} = [2]\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{0}^{\pi}\left[\frac{81}{4}\right] = \frac{81\pi}{4}$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{81\pi}{4 \cdot 36} = \frac{9\pi}{16}$$

5. Which of the following integrals will give the volume of the donut given in spherical coordinates by $\rho = \sin\varphi$.



a. $\int_0^{\pi}\int_0^{2\pi}\int_0^{\sin\varphi} \rho^2 \cos\varphi \, d\rho \, d\varphi \, d\theta$

b. $\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \sin\varphi \, d\rho \, d\varphi \, d\theta$

c. $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sin\varphi \, \rho^2 \cos\varphi \, d\rho \, d\varphi \, d\theta$

d. $\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$ correctchoice

e. $\int_0^{\pi}\int_0^{2\pi}\int_0^{\sin\varphi} 1 \, d\rho \, d\varphi \, d\theta$

$$V = \iiint 1 \, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

6. For the vector field $\vec{F} = (x^3, y^3, z^3)$, compute $\iiint \vec{\nabla} \cdot \vec{F} \, dV$ over the solid sphere $x^2 + y^2 + z^2 \le 4$.

a. $\dfrac{384}{5}\pi$ correctchoice

b. $8\pi^2$

c. $16\pi^2$

d. $64\pi^2$

e. 0

$$\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$$

$$\iiint \vec{\nabla} \cdot \vec{F} \, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 3\rho^2 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} 1 \, d\theta \int_0^{\pi} \sin\varphi \, d\varphi \int_0^{2} 3\rho^4 \, d\rho$$

$$= [2\pi]\Big[-\cos\varphi\Big]_0^{\pi}\left[\frac{3\rho^5}{5}\right]_0^2 = [2\pi][2]\left[\frac{3 \cdot 2^5}{5}\right] = \frac{3 \cdot 2^7 \pi}{5} = \frac{384}{5}\pi$$

7. For the vector field $\vec{F} = (-yz^2, xz^2, z^3)$, compute $\oint \vec{F} \cdot d\vec{s}$ once around the circle $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4)$.

a. 2π

b. 4π

c. 16π

d. 64π

e. 128π correctchoice

$\vec{F}(\vec{r}(\theta)) = (-32\sin\theta, 32\cos\theta, 64) \qquad \vec{v} = (-2\sin\theta, 2\cos\theta, 0)$

$\oint \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot \vec{v}\, d\theta = \int_0^{2\pi} (64\sin^2\theta + 64\cos^2\theta)\, d\theta = \int_0^{2\pi} 64\, d\theta = 128\pi$

8. Compute $\int_P^Q \vec{F} \cdot d\vec{s}$ along the straight line segment from $P = (1,2,4)$ to $Q = (2,-1,3)$ if $\vec{F} = (yz, xz, xy)$.

a. -14 correctchoice
b. -2

c. 2

d. 7

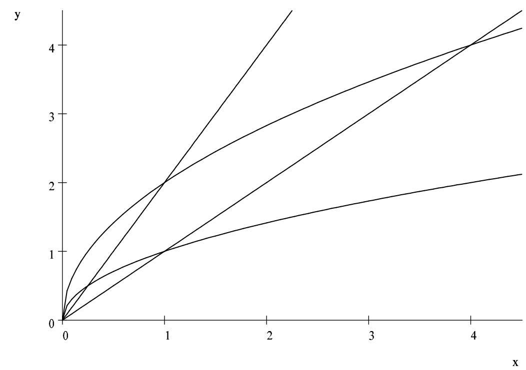
e. 14

Since $\vec{F} = \vec{\nabla} f$ where $f = xyz$, by the Fundamental Theorem of Calculus for Curves we have $\int_P^Q \vec{F} \cdot d\vec{s} = \int_P^Q \vec{\nabla} f \cdot d\vec{s} = f(Q) - f(P) = (2)(-1)(3) - (1)(2)(4) = -14$

9. (10 points) Compute $\iint \frac{1}{x^2}\,dx\,dy$ over the diamond shaped region bounded by the curves

$$y = \sqrt{x}\,, \quad y = 2\sqrt{x}\,, \quad y = x \quad \text{and} \quad y = 2x.$$

HINT: Let $u = \frac{y^2}{x}$ and $v = \frac{y}{x}$.



We solve for x and y so we can compute the Jacobian:

$$\frac{u}{v} = \frac{y^2}{x}\frac{x}{y} = y \qquad x = \frac{y}{v} = \frac{u}{v^2} \qquad \text{So } x = \frac{u}{v^2} \qquad y = \frac{u}{v}$$

$$J = \left\| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right\| = \left\| \begin{vmatrix} \frac{1}{v^2} & \frac{-2u}{v^3} \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} \right\| = \left| \frac{-u}{v^4} - \frac{-2u}{v^4} \right| = \frac{u}{v^4}$$

The boundaries are: $y^2 = x$ or $u = 1$. $y^2 = 4x$ or $u = 4$. $y = x$ or $v = 1$. $y = 2x$ or $v = 2$.

The integrand is: $\frac{1}{x^2} = \frac{v^4}{u^2}$ So

$$\iint \frac{1}{x^2}\,dx\,dy = \int_1^2 \int_1^4 \frac{v^4}{u^2} \cdot \frac{u}{v^4}\,du\,dv = \int_1^2 dv \int_1^4 \frac{1}{u}\,du = \Big[v\Big]_1^2 \Big[\ln|u|\Big]_1^4 = [2-1][\ln 4 - \ln 1] = \ln 4$$

10. (10 points) Find the area of the parametric surface $\vec{R}(u,v) = (u,v,uv)$ for $u^2 + v^2 \le 3$.

$$\vec{e}_u = (\ 1\ ,\ 0\ ,\ v\ )$$

$$\vec{e}_v = (\ 0\ ,\ 1\ ,\ u\ )$$

$$\vec{N} = \hat{\imath}(-v) - \hat{\jmath}(u) + \hat{k}(1) = (-v,-u,1)$$

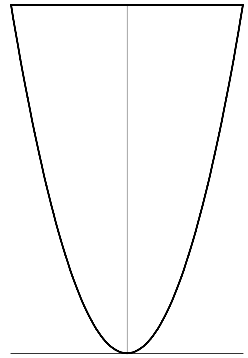
$$\left|\vec{N}\right| = \sqrt{v^2 + u^2 + 1}$$

$$A = \iint \left|\vec{N}\right| du\,dv = \iint \sqrt{v^2 + u^2 + 1}\,du\,dv$$

Switch to polar coordinates:

$$A = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{r^2+1}\; r\,dr\,d\theta = 2\pi \frac{(r^2+1)^{3/2}}{3} \Bigg|_0^{\sqrt{3}} = \frac{2\pi}{3}(4^{3/2} - 1^{3/2}) = \frac{2\pi}{3}(8-1) = \frac{14\pi}{3}$$

11. (20 points) Compute $\oint (x^2 y)\,dx + (x^3)\,dy$ counterclockwise around the boundary of the region between the parabola $y = x^2$ and the line $y = 9$ in two ways:



a. Directly by parametrizing the curves.

P: $\vec{r}(t) = (t, t^2) \qquad -3 \le t \le 3 \qquad \vec{v} = (1, 2t) \qquad \vec{F} = (x^2 y, x^3) = (t^4, t^3)$

$$\int_P \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot \vec{v}\,dt = \int_{-3}^{3} 3t^4\,dt = \frac{3t^5}{5}\Big|_{-3}^{3} = \frac{2 \cdot 3^6}{5}$$

L: $\vec{r}(t) = (-t, 9) \qquad -3 \le t \le 3 \qquad \vec{v} = (-1, 0) \qquad \vec{F} = (x^2 y, x^3) = (9t^2, -t^3)$

$$\int_L \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot \vec{v}\,dt = \int_{-3}^{3} -9t^2\,dt = -3t^3\Big|_{-3}^{3} = -2 \cdot 3^4$$

$$\oint (x^2 y)\,dx + (x^3)\,dy = \int_P \vec{F} \cdot d\vec{s} + \int_L \vec{F} \cdot d\vec{s} = \frac{2 \cdot 3^6}{5} - 2 \cdot 3^4 = 2 \cdot 3^4\left(\frac{9}{5} - 1\right) = \frac{8 \cdot 3^4}{5} = \frac{648}{5}$$

b. By using Green's theorem: $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_{\partial R} P\,dx + Q\,dy.$

$P = x^2 y \qquad Q = x^3 \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 - x^2 = 2x^2$

$$\oint (x^2 y)\,dx + (x^3)\,dy = \int_{-3}^{3} \int_{x^2}^{9} 2x^2\,dy\,dx = \int_{-3}^{3} \left[2x^2 y\right]_{x^2}^{9} dx = \int_{-3}^{3} (18x^2 - 2x^4)\,dx$$

$$= \left[6x^3 - \frac{2x^5}{5}\right]_{-3}^{3} = 2\left[2 \cdot 3^4 - \frac{2 \cdot 3^5}{5}\right] = 4 \cdot 3^4\left[1 - \frac{3}{5}\right] = \frac{8 \cdot 3^4}{5} = \frac{648}{5}$$

12. (20 points) For the vector field $\vec{F} = (-yz^2, xz^2, z^3)$, compute $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the paraboloid $z = x^2 + y^2$ for $z \le 4$ with normal pointing down and out. Use the following steps: The surface may be parametrized by

$$\vec{R}(r,\theta) = (\quad r\cos\theta \quad , \quad r\sin\theta \quad , \quad r^2 \quad)$$

a. Find the tangent vectors:

$$\vec{e}_r = (\quad \cos\theta \quad , \quad \sin\theta \quad , \quad 2r \quad)$$

$$\vec{e}_\theta = (\quad -r\sin\theta \quad , \quad r\cos\theta \quad , \quad 0 \quad)$$

b. Find the normal vector:

$$\vec{N} = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$
$$= (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

Reverse orientation:

$$\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$$

c. Compute the curl of $\vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz^2 & xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0 - 2xz) - \hat{\jmath}(0 + 2yz) + \hat{k}(z^2 + z^2) = (-2xz, -2yz, 2z^2)$$

d. Evaluate $\vec{\nabla} \times \vec{F}$ on the surface:

$$\vec{\nabla} \times \vec{F}\left(\vec{R}(r,\theta)\right) = (-2r^3\cos\theta, -2r^3\sin\theta, 2r^4)$$

e. Compute the integral:

$$\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (-4r^5\cos^2\theta - 4r^5\sin^2\theta - 2r^5)\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^2 (-6r^5)\, dr\, d\theta = -2\pi\left[r^6\right]_0^2 = -128\pi$$