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1-10	/50	12	/20
11	/10	13	/20

MATH 253 Final Exam Fall 2003

Sections 504-506 Solutions P. Yasskin

Multiple Choice: (5 points each) Work Out: (points indicated)

1. Find a parametric equation of the line tangent to the curve $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, \theta)$ at the point $(-2,0,\pi)$.

a. $X(t) = (-2t, -2, 1 + \pi t)$

b. $X(t) = (-2, -2t, \pi + t)$ correctchoice

c. $X(t) = (-2t - 2, 0, \pi + t)$

d. $X(t) = (0, -2t - 2, \pi + t)$

e. $X(t) = (-2t - 2, 0, 1 + \pi t)$

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, \theta) = (-2,0,\pi)$ at $\theta = \pi$.
$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 1)$ $\vec{v}(\pi) = (0,-2,1)$
$X(t) = (-2,0,\pi) + t(0,-2,1) = (-2,-2t,\pi + t)$

2. The density of the fog is given by $\rho = 30 - x^2 - y^2 - z$. If an airplane is at the position $(x,y,z) = \left(2, \sqrt{2}, 4\right)$, in what unit vector direction should the airplane initially travel to get out of the fog as quickly as possible?

a. $\left(\frac{-2}{\sqrt{10}}, \frac{-\sqrt{2}}{\sqrt{10}}, \frac{-2}{\sqrt{10}}\right)$

b. $\left(\frac{2}{\sqrt{10}}, \frac{\sqrt{2}}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$

c. $\left(\frac{2}{\sqrt{10}}, \frac{-\sqrt{2}}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$

d. $\left(\frac{-4}{5}, \frac{-2\sqrt{2}}{5}, \frac{-1}{5}\right)$

e. $\left(\frac{4}{5}, \frac{2\sqrt{2}}{5}, \frac{1}{5}\right)$ correctchoice

$\vec{\nabla}\rho = (-2x, -2y, -1)$ $\vec{\nabla}\rho\left(2,\sqrt{2},4\right) = \left(-4, -2\sqrt{2}, -1\right)$
To decrease the density, the airplane should travel in the direction $-\vec{\nabla}\rho\left(2,\sqrt{2},4\right) = \left(4, 2\sqrt{2}, 1\right)$.
Since $\left|-\vec{\nabla}\rho\right| = \sqrt{16 + 8 + 1} = 5$, the unit vector direction is $\left(\frac{4}{5}, \frac{2\sqrt{2}}{5}, \frac{1}{5}\right)$.

3. Find a parametric equation of the plane tangent to the surface $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$ at the point where $(r,\theta) = (2,\pi)$.

a. $X(s,t) = (-1,0,0) + s(-2,0,\pi) + t(0,-2,1)$

b. $X(s,t) = (1,0,0) + s(2,0,\pi) + t(0,2,1)$

c. $X(s,t) = (0,-2,1) + s(-1,0,0) + t(-2,0,\pi)$

d. $X(s,t) = (-2,0,\pi) + s(-1,0,0) + t(0,-2,1)$ correctchoice

e. $X(s,t) = (2,0,\pi) + s(1,0,0) + t(0,2,1)$

$\vec{R}(2,\pi) = (2\cos\pi, 2\sin\pi, \pi) = (-2,0,\pi)$

$\vec{e}_r = (\cos\theta, \sin\theta, 0) \qquad \vec{e}_r(2,\pi) = (\cos\pi, \sin\pi, 0) = (-1,0,0)$

$\vec{e}_\theta = (-r\sin\theta, r\cos\theta, 1) \qquad \vec{e}_\theta(2,\pi) = (-2\sin\pi, 2\cos\pi, 1) = (0,-2,1)$

$X(s,t) = \vec{R}(2,\pi) + s\vec{e}_r(2,\pi) + t\vec{e}_\theta(2,\pi) = (-2,0,\pi) + s(-1,0,0) + t(0,-2,1)$

4. Find the non-parametric equation of the plane tangent to the surface $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$ at the point where $(r,\theta) = (2,\pi)$.

a. $y + 2z = 2\pi$ correctchoice

b. $-x = -2y + z$

c. $-x + 2y - z = -2 + \pi$

d. $-x - 2y - z = -2 + \pi$

e. $-y + 2z = 2\pi$

From #3, $P = \vec{R}(2,\pi) = (-2,0,\pi)$, $\vec{e}_r = (-1,0,0)$ and $\vec{e}_\theta = (0,-2,1)$. So

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & 0 \\ 0 & -2 & 1 \end{vmatrix} = \hat{\imath}(0) - \hat{\jmath}(-1) + \hat{k}(2) = (0,1,2) \qquad \vec{N}\cdot X = \vec{N}\cdot P \qquad y + 2z = 2\pi$$

5. Find the non-parametric equation of the plane tangent to the surface $x^2y^2 + x^2z^2 + y^2z^2 = 49$ at the point $(1,2,3)$.

a. $13x + 10y + 5z = 48$

b. $13x - 10y + 5z = 8$

c. $13x + 20y + 15z = 98$ correctchoice

d. $13x - 20y + 15z = 18$

e. $39x + 20y + 5z = 94$

$f = x^2y^2 + x^2z^2 + y^2z^2 \qquad P = (1,2,3)$

$\vec{\nabla}f = (2xy^2 + 2xz^2, 2yx^2 + 2yz^2, 2zx^2 + 2zy^2) \qquad \vec{N} = \vec{\nabla}f(P) = (26,40,30)$

$\vec{N}\cdot X = \vec{N}\cdot P \qquad 26x + 40y + 30z = 26\cdot 1 + 40\cdot 2 + 30\cdot 3 = 196 \qquad 13x + 20y + 15z = 98$

6. Find the non-parametric equation of the plane tangent to the graph of the function $z = x^2y + xy^2$ at the point $(2,1)$.

a. $z = 5x + 8y + 6$

b. $z = -5x - 8y + 6$

c. $z = 5x + 8y - 12$ correctchoice

d. $z = -5x - 8y + 24$

e. $z = 5x + 8y - 6$

$f(x,y) = x^2y + xy^2 \qquad f_x(x,y) = 2xy + y^2 \qquad f_y(x,y) = x^2 + 2xy$

$f(2,1) = 6 \qquad f_x(2,1) = 5 \qquad f_y(2,1) = 8$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 6 + 5(x-2) + 8(y-1) = 5x + 8y - 12$

7. The point $(1,1)$ is a critical point of the function $f = 4xy - x^4 - y^4$. Using the Second Derivative Test, we find

a. $(1,1)$ is a local maximum. correctchoice

b. $(1,1)$ is a local minimum.

c. $(1,1)$ is an inflection point.

d. $(1,1)$ is a saddle point.

e. the test fails.

$f_x = 4y - 4x^3 \qquad f_y = 4x - 4y^3 \qquad f_{xx} = -12x^2 \qquad f_{yy} = -12y^2 \qquad f_{xy} = 4 \qquad D = 144x^2y^2 - 16$

$D(1,1) = 144 - 16 = 128 > 0 \qquad f_{xx}(1,1) = -12 < 0$ So $(1,1)$ is a local maximum.

8. Compute $\int_P^Q 2x\,dx + 2y\,dy + 2z\,dz$ along the straight line from $P = (1,2,2)$ to $Q = (3,4,12)$.

a. -10

b. $\sqrt{10}$

c. 10

d. 108

e. 160 correctchoice

Let $\vec{F} = (2x, 2y, 2z)$. Then $\vec{F} = \vec{\nabla} f$ where $f = x^2 + y^2 + z^2$. So by the FTCC,

$$\int_P^Q 2x\,dx + 2y\,dy + 2z\,dz = \int_P^Q \vec{F}\,d\vec{s} = \int_P^Q \vec{\nabla} f\,d\vec{s} = f(Q) - f(P) = (9 + 16 + 144) - (1 + 4 + 4) = 160$$

9. Compute $\oint 2x\,dx + 2xy\,dy$ counterclockwise around the boundary of the rectangle $1 \le x \le 4$, $2 \le y \le 4$.

a. 6

b. 18

c. 24

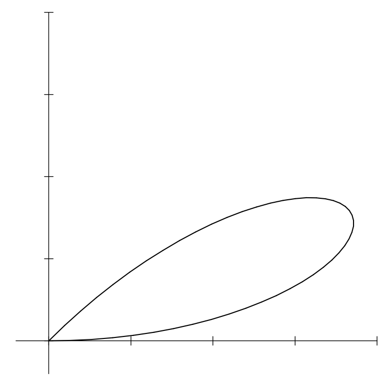
d. 36 correctchoice

e. 72

By Green's Theorem:

$$\oint 2x\,dx + 2xy\,dy = \int_2^4 \int_1^4 \partial_x(2xy) - \partial_y(2x)\,dx\,dy = \int_2^4 \int_1^4 2y\,dx\,dy = \int_1^4 1\,dx \int_2^4 2y\,dy = 3\left[y^2\right]_2^4 = 36$$

10. Find the area of one petal of the 4 leaf rose $r = \sin(4\theta)$.
The petal in the first quadrant.is shown.



a. $\frac{\pi}{16}$ correctchoice

b. $\frac{\pi}{8}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{2}$

e. π

$r = \sin(4\theta) = 0$ at $4\theta = \pi$ or $\theta = \frac{\pi}{4}$

$$A = \iint 1\,dA = \int_0^{\pi/4} \int_0^{\sin(4\theta)} r\,dr\,d\theta = \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{\sin(4\theta)} d\theta = \frac{1}{2}\int_0^{\pi/4} \sin^2(4\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1 - \cos(8\theta)}{2}\,d\theta = \frac{1}{4}\left[\theta - \frac{\sin(8\theta)}{8}\right]_0^{\pi/4} = \frac{\pi}{16}$$

11. (10 points) Find the point in the first octant on the graph of $xy^2z^4 = 32$ which is closest to the origin.
HINTS: What is the square of the distance from a point to the origin? Lagrange multipliers are easier.

Minimize $f = x^2 + y^2 + z^2$ subject to $g = xy^2z^4 = 32$.

Method 1: Lagrange multipliers:

$$\vec{\nabla}f = (2x, 2y, 2z) \qquad \vec{\nabla}g = (y^2z^4, 2xyz^4, 4xy^2z^3)$$

$$\vec{\nabla}f = \lambda\vec{\nabla}g \quad \Rightarrow \quad 2x = \lambda y^2z^4, \qquad 2y = \lambda 2xyz^4, \qquad 2z = \lambda 4xy^2z^3$$

$$\lambda = \frac{2x}{y^2z^4} = \frac{1}{xz^4} = \frac{1}{2xy^2z^2} \quad \Rightarrow \quad 2x^2 = y^2, \qquad 4x^2 = z^2 \quad \Rightarrow \quad y = \sqrt{2}\,x, \qquad z = 2x$$

$$32 = xy^2z^4 = x\left(\sqrt{2}\,x\right)^2(2x)^4 = 32x^7 \quad \Rightarrow \quad x = 1 \qquad y = \sqrt{2} \qquad z = 2$$

Method 2: Eliminate a variable:

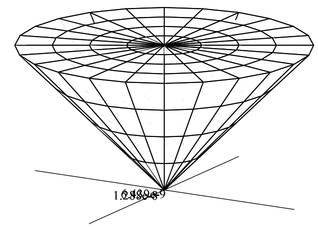
$$x = \frac{32}{y^2z^4} \qquad f = \frac{2^{10}}{y^4z^8} + y^2 + z^2$$

$$f_y = -\frac{2^{12}}{y^5z^8} + 2y = 0 \qquad f_z = -\frac{2^{13}}{y^4z^9} + 2z = 0 \quad \Rightarrow \quad y^6z^8 = 2^{11} \qquad y^4z^{10} = 2^{12}$$

$$\Rightarrow \quad 2 = \frac{y^4z^{10}}{y^6z^8} = \frac{z^2}{y^2} \quad \Rightarrow \quad z = \sqrt{2}\,y \quad \Rightarrow \quad y^6\left(\sqrt{2}\,y\right)^8 = 2^{11} \quad \Rightarrow \quad y^{14} = 2^7$$

$$\Rightarrow \quad y = \sqrt{2} \qquad z = 2 \qquad x = \frac{32}{y^2z^4} = \frac{2^5}{2 \cdot 2^4} = 1$$

12. (20 points) Verify Gauss' Theorem $\displaystyle\iiint_V \vec{\nabla} \cdot \vec{F}\,dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$



for the vector field $\vec{F} = (xy^2, yx^2, z^3)$ and the volume above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$.
Use the following steps:

a. Compute the volume integral:

$$\vec{\nabla} \cdot \vec{F} = y^2 + x^2 + 3z^2 = r^2 + 3z^2$$

$$\iiint_V \vec{\nabla} \cdot \vec{F}\,dV = \int_0^{2\pi}\int_0^2\int_r^2 (r^2 + 3z^2)r\,dz\,dr\,d\theta = 2\pi\int_0^2 \left[r^3z + z^3r\right]_{z=r}^2 dr = 2\pi\int_0^2 (2r^3 + 8r) - (2r^4)\,dr$$

$$= 2\pi\left[\frac{r^4}{2} + 4r^2 - \frac{2r^5}{5}\right]_0^2 = 2^5\pi\left(\frac{1}{2} + 1 - \frac{4}{5}\right) = 2^5\pi\left(\frac{5 + 10 - 8}{10}\right) = \frac{7 \cdot 2^4\pi}{5} = \frac{112}{5}\pi$$

b. Compute the surface integral over the cone using the parametrization

$\vec{R}(r,\theta) = (\quad r\cos\theta \quad , \quad r\sin\theta \quad , \quad r \quad)$:

$\vec{e}_r = (\quad \cos\theta \quad , \quad \sin\theta \quad , \quad 1 \quad)$

$\vec{e}_\theta = (\quad -r\sin\theta \quad , \quad r\cos\theta \quad , \quad 0 \quad)$

$\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$

Reverse orientation

$\vec{N} = (r\cos\theta, r\sin\theta, -r)$

$\vec{F}\left(\vec{R}(r,\theta)\right) = (xy^2, yx^2, z^3) = (r^3\cos\theta\sin^2\theta, r^3\sin\theta\cos^2\theta, r^3)$

$$\iint_C \vec{F}\cdot d\vec{S} = \iint_C \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (r^4\cos^2\theta\sin^2\theta + r^4\sin^2\theta\cos^2\theta - r^4)\,dr\,d\theta$$

$$= \int_0^2 r^4\,dr\int_0^{2\pi}(2\sin^2\theta\cos^2\theta - 1)\,d\theta = \left[\frac{r^5}{5}\right]_0^2\left(2\frac{\pi}{4} - 2\pi\right) = -\frac{48}{5}\pi$$

c. Compute the surface integral over the disk using the parametrization

$\vec{R}(r,\theta) = (\quad r\cos\theta \quad , \quad r\sin\theta \quad , \quad 2 \quad)$:

$\vec{e}_r = (\quad \cos\theta \quad , \quad \sin\theta \quad , \quad 0 \quad)$

$\vec{e}_\theta = (\quad -r\sin\theta \quad , \quad r\cos\theta \quad , \quad 0 \quad)$

$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (0, 0, r)$

$\vec{F}\left(\vec{R}(r,\theta)\right) = (xy^2, yx^2, z^3) = (r^3\cos\theta\sin^2\theta, r^3\sin\theta\cos^2\theta, 8)$

$$\iint_D \vec{F}\cdot d\vec{S} = \iint_C \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^2 8r\,dr\,d\theta = 2\pi\left[4r^2\right]_0^2 = 32\pi$$

d. Compute the surface integral over the total boundary:

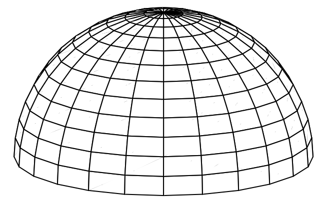
$$\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_C \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S} = -\frac{48}{5}\pi + 32\pi = \frac{112}{5}\pi$$

13. (20 points) Verify Stokes' Theorem $\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial H} \vec{F} \cdot d\vec{s}$

for the vector field $\vec{F} = (y, -x, xz + yz)$

and the hemisphere $z = \sqrt{9 - x^2 - y^2}$.

Use the following steps:



a. Parametrize the boundary curve and compute the line integral:

$$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 0)$$

$$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$$

$$\vec{F}(\vec{r}(\theta)) = (3\sin\theta, -3\cos\theta, 0)$$

$$\oint_{\partial H} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -9\sin^2\theta - 9\cos^2\theta\, d\theta = -9\int_0^{2\pi} d\theta = -18\pi$$

b. Parametrize the surface and compute the surface integral:

$$\vec{R}(\varphi, \theta) = (\ 3\sin\varphi\cos\theta,\quad 3\sin\varphi\sin\theta,\quad 3\cos\varphi\ )$$

$$\vec{e}_\varphi = (\ 3\cos\varphi\cos\theta,\quad 3\cos\varphi\sin\theta,\quad -3\sin\varphi\ )$$

$$\vec{e}_\theta = (\ -3\sin\varphi\sin\theta,\quad 3\sin\varphi\cos\theta,\quad 0\ )$$

$$\vec{N} = \hat{\imath}(9\sin^2\varphi\cos\theta) - \hat{\jmath}(-9\sin^2\varphi\sin\theta) + \hat{k}(9\sin\varphi\cos\varphi\cos^2\theta + 9\sin\varphi\cos\varphi\sin^2\theta)$$
$$= (9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & xz + yz \end{vmatrix} = \hat{\imath}(z) - \hat{\jmath}(z) + \hat{k}(-1 - 1) = (z, -z, -2) = (3\cos\varphi, -3\cos\varphi, -2)$$

$$\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_H \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^{\pi/2} (27\sin^2\varphi\cos\varphi\cos\theta - 27\sin^2\varphi\cos\varphi\sin\theta - 18\sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} (-18\sin\varphi\cos\varphi)\, d\varphi\, d\theta \qquad \text{because } \int_0^{2\pi} \cos\theta\, d\theta = \int_0^{2\pi} \sin\theta\, d\theta = 0$$

$$= 2\pi \int_0^{\pi/2} (-18\sin\varphi\cos\varphi)\, d\varphi = -36\pi \left[\frac{\sin^2\varphi}{2} \right]_0^{\pi/2} = -18\pi$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sin^2 A = \frac{1 - \cos(2A)}{2} \qquad \cos^2 A = \frac{1 + \cos(2A)}{2}$$

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \pi \qquad \int_0^{2\pi} \cos^2\theta\, d\theta = \pi$$

$$\int_0^{2\pi} \sin^4\theta\, d\theta = \frac{3}{4}\pi \qquad \int_0^{2\pi} \cos^4\theta\, d\theta = \frac{3}{4}\pi \qquad \int_0^{2\pi} \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{4}\pi$$