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MATH 253 Exam 1 Spring 2004

Sections 504-506 Solutions P. Yasskin

On the front of the Blue Book, on the Scantron and on this sheet
write your Name, your University ID, your Section and "Exam 1."
On the front of the Blue Book copy the Grading Grid shown at the right.
Enter your Multiple Choice answers on the Scantron
and CIRCLE them on this sheet.

Multiple Choice: (6 points each. No part credit.)

1. Find the area of the triangle with vertices $A = (1,1,1)$, $B = (3,1,4)$ and $C = (2,0,3)$.

a. 1

b. $2\sqrt{3}$

c. $\sqrt{3}$

d. $\sqrt{14}$

e. $\frac{1}{2}\sqrt{14}$ correctchoice

$\overrightarrow{AB} = B - A = (3,1,4) - (1,1,1) = (2,0,3)$ $\quad$ $\overrightarrow{AC} = C - A = (2,0,3) - (1,1,1) = (1,-1,2)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{vmatrix} = \hat{\imath}(3) - \hat{\jmath}(1) + \hat{k}(-2) = (3,-1,-2)$$

Area $= \frac{1}{2}\sqrt{9+1+4} = \frac{1}{2}\sqrt{14}$

2. Find the intersection of the line $\vec{r}(t) = (1+2t, 3+4t, 5+6t)$ and the plane $5x + 4y + 3z = 54$.

a. $(2,4,6)$

b. $(1,3,5)$

c. $(2,5,8)$ correctchoice

d. $(-2,4,-2)$

e. $(2,4,2)$

Substitute the line $x = 1+2t$, $y = 3+4t$, $z = 5+6t$ into the plane $5x+4y+3z = 54$

$5(1+2t) + 4(3+4t) + 3(5+6t) = 54 \quad \Rightarrow \quad 32 + 44t = 54 \quad \Rightarrow \quad 44t = 22 \quad \Rightarrow \quad t = \frac{1}{2}$

$\vec{r}\left(\frac{1}{2}\right) = \left(1 + 2\cdot\frac{1}{2}, 3 + 4\cdot\frac{1}{2}, 5 + 6\cdot\frac{1}{2}\right) = (2,5,8)$

3. A satellite is orbiting from East to West directly above the equator. In which direction does the binormal $\hat{B}$ point?

a. North

b. South correctchoice

c. Up

d. Down

e. West

$\hat{T}$ points West. $\hat{N}$ points Down toward the center of the earth. So $\hat{B} = \hat{T} \times \hat{N}$ points South.

4. Find the arclength of the parametric curve $\vec{r}(t) = (2t^2, t^2, t^3)$ between the points $(0,0,0)$ and $(2,1,1)$.

a. $\frac{1}{27}(29^{3/2} - 20^{3/2})$ correctchoice

b. $12(29^{3/2} - 20^{3/2})$

c. $12(56^{3/2} - 20^{3/2})$

d. $\frac{1}{12}(56^{3/2} - 20^{3/2})$

e. $\frac{1}{27}(56^{3/2} - 29^{3/2})$

$\vec{v} = (4t, 2t, 3t^2) \qquad |\vec{v}| = \sqrt{16t^2 + 4t^2 + 9t^4} = t\sqrt{20 + 9t^2}$

$L = \int_{(0,0,0)}^{(2,1,1)} ds = \int_0^1 |\vec{v}|\,dt = \int_0^1 t\sqrt{20 + 9t^2}\,dt \qquad u = 20 + 9t^2 \qquad du = 18t\,dt$

$L = \frac{1}{18}\int_{20}^{29} \sqrt{u}\,du = \frac{1}{18}\frac{2u^{3/2}}{3}\Big|_{20}^{29} = \frac{1}{27}(29^{3/2} - 20^{3/2})$

5. If $f(x,y) = g(x)h(y)$, then

a. $f_{xy}(x,y) = g'(x)h'(y)$ correctchoice

b. $f_{xy}(x,y) = g'(x)h(y) + g(x)h'(y)$

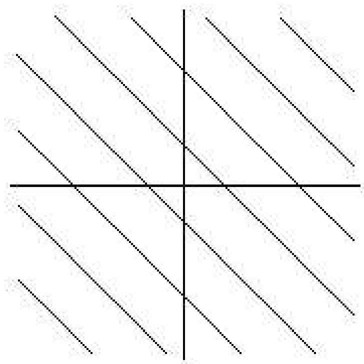
c. $f_{xy}(x,y) = g'(x)h(y) - g(x)h'(y)$

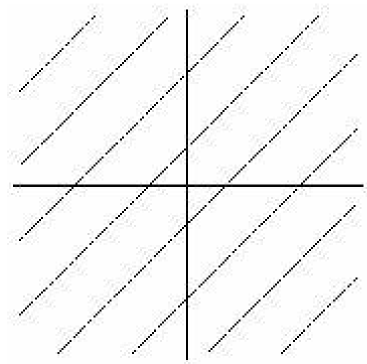
d. $f_{xy}(x,y) = g'(y)h(x) + g(y)h'(x)$

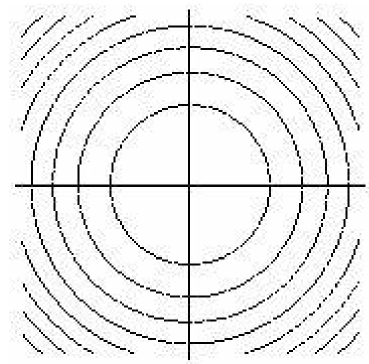
e. $f_{xy}(x,y) = 0$

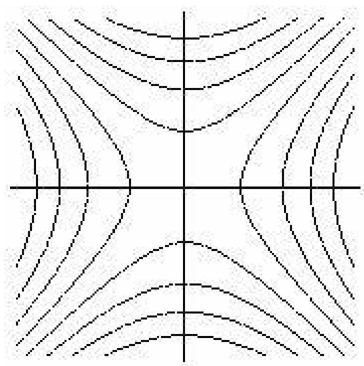
$f_x(x,y) = g'(x)h(y)$ ($h(y)$ is a constant) $f_{xy}(x,y) = g'(x)h'(y)$ (Now $g'(x)$ is a constant)

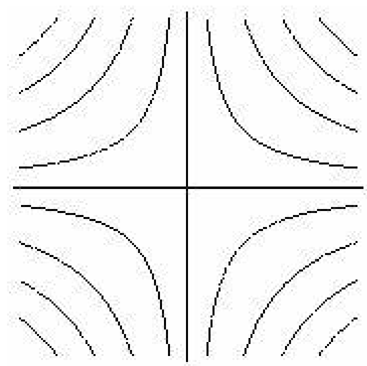
6. Which of the following is the contour plot of $f(x,y) = x^2 - y^2$?

a. 

b. 

c. 

d. 

e. 

$x^2 - y^2 = C$ is a hyperbola opening left and right or up and down. (d)

7. If $g(x,y) = x^3 \cos(xy)$, find $\dfrac{\partial g}{\partial x}$.

a. $3x^2 \cos(xy) + x^3 y \sin(xy)$

b. $3x^2 \cos(xy) - x^3 y \sin(xy)$ correctchoice

c. $-3x^2 \sin(y)$

d. $-3x^2 y \sin(xy)$

e. $-3x^2 \sin(xy)$

Product Rule: $\dfrac{\partial g}{\partial x} = -x^3 \sin(xy)y + \cos(xy)3x^2$

8. If $g(x,y) = x^3 \cos(xy)$, find $\dfrac{\partial^3 g}{\partial x \partial y^2}$.

a. $-5x^4 \sin(xy) + x^5 y \cos(xy)$

b. $5x^4 \sin(xy) - x^5 y \cos(xy)$

c. $-5x^4 \cos(xy) + x^5 y \sin(xy)$ correctchoice

d. $5x^4 \cos(xy) - x^5 y \sin(xy)$

e. $5x^4 \sin(xy) + x^5 y \cos(xy)$

$\dfrac{\partial g}{\partial y} = -x^4 \sin(xy)$ $\quad \dfrac{\partial^2 g}{\partial y^2} = -x^5 \cos(xy)$ $\quad \dfrac{\partial^3 g}{\partial x \partial y^2} = -5x^4 \cos(xy) + x^5 y \sin(xy)$

Work Out: (15 points each. Part credit possible.)

Start each problem on a new page of the Blue Book. Number the problem. Show all work.

9. (15 points) Consider the two parametric lines $\vec{r}_1(s) = (s, 5-s, 6-2s)$ and $\vec{r}_2(t) = (4-2t, 2+t, 3-t)$.

a. Find the point where the two lines intersect.

Equate components: $s = 4-2t, \quad 5-s = 2+t, \quad 6-2s = 3-t$
Substitute the first equation into the other two equations:
$5-(4-2t) = 2+t \quad \Rightarrow \quad t = 1, \quad s = 2$
$6-2(4-2t) = 3-t \quad \Rightarrow \quad 5t = 5 \quad \Rightarrow \quad t = 1, \quad s = 2$
Substitute into the two curves
$\vec{r}_1(2) = (2, 5-2, 6-2\cdot 2) = (2,3,2) \qquad \vec{r}_2(1) = (4-2, 2+1, 3-1) = (2,3,2)$
So the curves intersect at the point $P = (2,3,2)$.

b. Find the normal equation of the plane containing the two lines.

The normal to the plane is $\vec{N} = \vec{v}_1 \times \vec{v}_2$ where $\vec{v}_1$ and $\vec{v}_2$ are the two tangent vectors. Thus,

$$\vec{v}_1 = (1,-1,-2) \qquad \vec{v}_2 = (-2,1,-1)$$

$$\vec{N} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & -2 \\ -2 & 1 & -1 \end{vmatrix} = \hat{\imath}(3) - \hat{\jmath}(-5) + \hat{k}(-1) = (3,5,-1)$$

A point on the plane may be taken as a point on either line, say $P = \vec{r}_1(0) = (0,5,6)$.
$\vec{N}\cdot X = \vec{N}\cdot P \qquad 3x+5y-z = 5\cdot 5 - 6 = 19$

10. (15 points) Consider the function $f(x,y) = e^{xy}$.

a. Find the equation of the plane tangent to the graph of $z = f(x,y)$ at the point $\left(2, \frac{1}{2}\right)$.

$$f = e^{xy} \qquad f\left(2,\frac{1}{2}\right) = e^1 = e$$

$$f_x = ye^{xy} \qquad f_x\left(2,\frac{1}{2}\right) = \frac{1}{2}e^1 = \frac{1}{2}e$$

$$f_y = xe^{xy} \qquad f_y\left(2,\frac{1}{2}\right) = 2e^1 = 2e$$

$$z = f\left(2,\frac{1}{2}\right) + f_x\left(2,\frac{1}{2}\right)(x-2) + f_y\left(2,\frac{1}{2}\right)\left(y-\frac{1}{2}\right) = e + \frac{1}{2}e(x-2) + 2e\left(y-\frac{1}{2}\right)$$

$$z = \frac{1}{2}ex + 2ey - e$$

b. Use the linear approximation to $f(x,y)$ to estimate $f(2.4, 0.6)$.
(Express the answer as a multiple of e.)

$$f_{\tan}(x,y) = e + \frac{1}{2}e(x-2) + 2e(y-.5)$$

$$f(2.4,0.6) \approx f_{\tan}(2.4,0.6) = e + \frac{1}{2}e(.4) + 2e(.1) == e + .2e + .2e = 1.4e$$

11. $(15$ points$)$ The surface $xz^2 + z\cos y = \frac{15}{2}$ implicitly defines z as a function of x and y.

Find $\frac{\partial z}{\partial y}$ at the point $(x,y,z) = \left(2, \frac{\pi}{6}, \sqrt{3}\right)$.

Note: $\sin\frac{\pi}{6} = \frac{1}{2}$ $\qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Apply $\frac{\partial}{\partial y}$ to both sides: $\quad x2z\frac{\partial z}{\partial y} + \left(-z\sin y + \frac{\partial z}{\partial y}\cos y\right) = 0$

Solve for $\frac{\partial z}{\partial y}$: $\quad \frac{\partial z}{\partial y}(x2z + \cos y) = z\sin y \qquad \frac{\partial z}{\partial y} = \frac{z\sin y}{x2z + \cos y}$

Evaluate: $\left.\frac{\partial z}{\partial y}\right|_{\left(2,\frac{\pi}{6},\sqrt{3}\right)} = \frac{\sqrt{3}\,\sin\frac{\pi}{6}}{2\cdot 2\cdot\sqrt{3} + \cos\frac{\pi}{6}} = \frac{\sqrt{3}\cdot\frac{1}{2}}{2\cdot 2\cdot\sqrt{3} + \frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{8\sqrt{3} + \sqrt{3}} = \frac{1}{9}$

12. $(15$ points$)$ Determine whether each of the following limits exists and say why or why not. If the limit exists, find it.

a. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^2)^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2y^2}{(x^2+y^2)^2} = \lim_{x\to 0}\frac{x^2m^2x^2}{(x^2+m^2x^2)^2} = \lim_{x\to 0}\frac{m^2}{(1+m^2)^2}$$

which is different for different m's.

So the limit does not exist.

b. $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{(x^2+y^2)^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=r\cos\theta,y=r\sin\theta}} \frac{x^2y^3}{(x^2+y^2)^2} = \lim_{r\to 0}\frac{r^2\cos^2\theta\, r^3\sin^3\theta}{(r^2)^2} = \lim_{r\to 0} r\cos^2\theta\sin^3\theta = 0$$

independent of the behavior of θ. So the limit exists and equals 0.