Name________________________ ID________________ Section______

MATH 253	Exam 3	Spring 2004
Sections 504-506	Solutions	P. Yasskin

1-6	/42
7	/15
8	/15
9	/30
Total	/102

On the front of the Blue Book, on the Scantron and on this sheet
write your Name, your University ID, your Section and "Exam 3."
On the front of the Blue Book copy the Grading Grid shown at the right.
Enter your Multiple Choice answers on the Scantron
and CIRCLE them on this sheet.

Multiple Choice: (7 points each. No part credit.)

1. Compute $\int_0^3 \int_0^2 (x+y)\,dx\,dy$

a. $\frac{13}{2}$

b. $\frac{21}{2}$

c. 11

d. 15 correctchoice

e. $\frac{41}{2}$

$$\int_0^3 \int_0^2 (x+y)\,dx\,dy = \int_0^3 \left[\frac{x^2}{2} + xy\right]_{x=0}^2 dy = \int_0^3 (2+2y)\,dy = \left[2y + y^2\right]_{y=0}^3 = 6+9 = 15$$

2. Compute $\iint_R y\,dA$ over the region in the first quadrant between the circles $x^2+y^2=4$ and $x^2+y^2=25$.

a. 0

b. 39 correctchoice

c. $\frac{125}{3}$

d. 78

e. 5187

$$\iint_R y\,dA = \int_0^{\pi/2} \int_2^5 r\sin\theta\, r\,dr\,d\theta = \left[-\cos\theta\right]_0^{\pi/2} \left[\frac{r^3}{3}\right]_2^5 = (1)\left(\frac{125-8}{3}\right) = 39$$

3. A metal plate covers the region between $y = x^2$ and $y = x$ and has surface density $\rho = y$. Find the mass of the plate.

a. $\frac{1}{2}$

b. $\frac{1}{3}$

c. $\frac{1}{4}$

d. $\frac{1}{6}$

e. $\frac{1}{15}$ correctchoice

$$M = \int\int \rho\, dA = \int_0^1 \int_{x^2}^x y\, dy\, dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^x dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_{x=0}^1 = \frac{1}{6} - \frac{1}{10} = \frac{5-3}{30} = \frac{2}{30} = \frac{1}{15}$$

4. A metal plate covers the region between $y = x^2$ and $y = x$ and has surface density $\rho = y$. Find the x-component of the center of mass of the plate.

a. $\frac{1}{24}$

b. $\frac{1}{12}$

c. $\frac{1}{2}$

d. $\frac{5}{8}$ correctchoice

e. $\frac{3}{4}$

$$x\text{-mom} = \int\int x\rho\, dA = \int_0^1 \int_{x^2}^x xy\, dy\, dx = \int_0^1 \left[x\frac{y^2}{2} \right]_{y=x^2}^x dx = \int_0^1 \left(\frac{x^3}{2} - \frac{x^5}{2} \right) dx$$

$$= \left[\frac{x^4}{8} - \frac{x^6}{12} \right]_{x=0}^1 = \frac{1}{8} - \frac{1}{12} = \frac{3-2}{24} = \frac{1}{24} \qquad \bar{x} = \frac{x\text{-mom}}{M} = \frac{1}{24}\,\frac{15}{1} = \frac{5}{8}$$

5. Find the volume of the region above the paraboloid $z = 2x^2 + 2y^2$ and below the paraboloid $z = 12 - x^2 - y^2$.
HINT: Where do the paraboloids intersect?

a. 24π correctchoice

b. 32π

c. 48π

d. 60π

e. 72π

The paraboloids intersect where:
$2x^2 + 2y^2 = 12 - x^2 - y^2 \quad \Rightarrow \quad 3x^2 + 3y^2 = 12 \quad \Rightarrow \quad x^2 + y^2 = 4$
The volume is
$$V = \iint_R [(12 - x^2 - y^2) - (2x^2 + 2y^2)]\, dA = \iint_R (12 - 3x^2 - 3y^2)\, dA$$
We convert to polar coordinates:
$$V = \int_0^{2\pi} \int_0^2 (12 - 3r^2)\, r\, dr\, d\theta = 2\pi\left[12\frac{r^2}{2} - 3\frac{r^4}{4}\right]_0^2 = 2\pi[24 - 12] = 24\pi$$

6. Compute $\int (yz\,dx + xz\,dy + xy\,dz)$ along the curve $\vec{r}(t) = (t^3, t^2, t)$ for $0 \le t \le 2$.

a. 4

b. 16

c. 64 correctchoice

d. 256

e. 1024

$dx = 3t^2\,dt \qquad dy = 2t\,dt \qquad dz = dt$
$$\int (yz\,dx + xz\,dy + xy\,dz) = \int_0^2 (t^3\,3t^2\,dt + t^4\,2t\,dt + t^5\,dt) = \int_0^2 6t^5\,dt = \left[t^6\right]_0^2 = 64$$

Work Out: (Part credit possible.)

Start each problem on a new page of the Blue Book. Number the problem. Show all work.

7. (15 points) Compute the integral

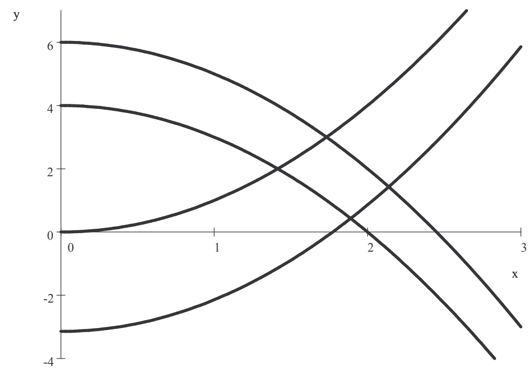
$$I = \int\int x\sin(x^2 - y)\,dx\,dy$$

over the "diamond" shaped region in the first quadrant between the curves

$y = x^2$, $y = x^2 - \pi$, $y = 4 - x^2$, and $y = 6 - x^2$.

HINT: Use curvilinear coordinates defined by

$$u = \frac{x^2 + y}{2} \quad \text{and} \quad v = \frac{x^2 - y}{2}.$$



Let $u = \frac{x^2 + y}{2}$ and $v = \frac{x^2 - y}{2}$. Then $u + v = x^2$ and $u - v = y$.

So $x = \sqrt{u + v}$ and $y = u - v$. The boundary curves are:

$y = x^2 \quad \Rightarrow \quad u - v = u + v \quad \Rightarrow \quad 2v = 0 \quad \Rightarrow \quad v = 0$

$y = x^2 - \pi \quad \Rightarrow \quad u - v = u + v - \pi \quad \Rightarrow \quad 2v = \pi \quad \Rightarrow \quad v = \pi/2$

$y = 4 - x^2 \quad \Rightarrow \quad u - v = 4 - u - v \quad \Rightarrow \quad 2u = 4 \quad \Rightarrow \quad u = 2$

$y = 6 - x^2 \quad \Rightarrow \quad u - v = 6 - u - v \quad \Rightarrow \quad 2u = 6 \quad \Rightarrow \quad u = 3$

We compute the Jacobian:

$$J = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{1}{2\sqrt{u+v}} & \frac{1}{2\sqrt{u+v}} \\ 1 & -1 \end{pmatrix}\right|$$

$$= \left|\frac{-1}{2\sqrt{u+v}} - \frac{1}{2\sqrt{u+v}}\right| = \left|\frac{-1}{\sqrt{u+v}}\right| = \frac{1}{\sqrt{u+v}}$$

The integrand is $x\sin(x^2 - y) = \sqrt{u+v}\,\sin(2v)$. So the integral is

$$I = \int\int x\sin(x^2 + y)\,dx\,dy = \int_0^{\pi/2}\int_2^3 \sqrt{u+v}\,\sin(2v)\frac{1}{\sqrt{u+v}}\,du\,dv = \int_0^{\pi/2}\int_2^3 \sin(2v)\,du\,dv$$

$$= \int_0^{\pi/2}\sin(2v)\,dv\int_2^3 1\,du = \left[\frac{-\cos(2v)}{2}\right]_0^{\pi/2}\Big[u\Big]_2^3 = \left(\frac{--1}{2} - \frac{-1}{2}\right)(3-2) = 1$$

8. (15 points) Consider the vector field $\vec{G} = (xz^2, yz^2, 2x^2z + 2y^2z)$.

a. Compute $\vec{\nabla}\cdot\vec{G}$.

$$\vec{\nabla}\cdot\vec{G} = \frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(2x^2z + 2y^2z) = z^2 + z^2 + 2x^2 + 2y^2 = 2(x^2 + y^2 + z^2)$$

b. Compute $\iiint \vec{\nabla}\cdot\vec{G}\,dV$ over the hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$.

In spherical coordinates $\vec{\nabla}\cdot\vec{G} = 2\rho^2$.

$$\iiint \vec{\nabla}\cdot\vec{G}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 2\rho^2\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\Big[-\cos\varphi\Big]_0^{\pi/2}\left[\frac{2\rho^5}{5}\right]_0^2 = 2\pi[1]\left[\frac{2^6}{5}\right] = \frac{2^7\pi}{5}$$

9. $\left(30 \text{ points}\right)$ The paraboloid $z = x^2 + y^2$ for $z \le 9$ may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

a. Find the area of the paraboloid, by computing each of the following:

$\vec{e}_r, \quad \vec{e}_\theta, \quad \vec{N}, \quad \left|\vec{N}\right|, \quad A = \iint dS$

HINT: Factor the quantity in the square root.

$\vec{e}_r = (\cos\theta, \sin\theta, 2r)$

$\vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$

$\left|\vec{N}\right| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$

$A = \iint dS = \iint \left|\vec{N}\right| dr\,d\theta = \int_0^{2\pi}\int_0^3 r\sqrt{4r^2+1}\,dr\,d\theta$ Let $u = 4r^2 + 1$, $du = 8r\,dr$

$A = \frac{2\pi}{8}\int_1^{37}\sqrt{u}\,du = \frac{\pi}{4}\left[\frac{2u^{3/2}}{3}\right]_1^{37} = \frac{\pi}{6}(37^{3/2} - 1)$

b. For the vector field $\vec{F} = (-yz, xz, z^2)$, compute $\vec{\nabla}\times\vec{F}$ and $\iint \vec{\nabla}\times\vec{F}\cdot d\vec{S}$ over the paraboloid oriented down and out. Be sure to check the orientation.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(-x) - \hat{\jmath}(0 - -y) + \hat{k}(z - -z) = (-x, -y, 2z)$$

$\vec{\nabla}\times\vec{F}\left(\vec{R}(r,\theta)\right) = (-r\cos\theta, -r\sin\theta, 2r^2)$

The normal points up and in, so reverse it: $\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$

$\iint \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \iint \vec{\nabla}\times\vec{F}\left(\vec{R}(r,\theta)\right)\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (-2r^3\cos^2\theta - 2r^3\sin^2\theta - 2r^3)\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^3 (-4r^3)\,dr\,d\theta = 2\pi\left[-r^4\right]_0^3 = -162\pi$

c. The boundary of the paraboloid is the circle $x^2 + y^2 = 9$, with $z = 9$, which may be parametrized by $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 9)$. Compute $\int \vec{F}\cdot d\vec{s}$ clockwise around this circle for $\vec{F} = (-yz, xz, z^2)$. Be sure to check the orientation.

NOTE: The answer to parts (b) and (c) should be equal by Stokes' Theorem.

$\vec{v} = (-3\sin\theta, 3\cos\theta, 0)$

This is counterclockwise, so reverse it: $\vec{v} = (3\sin\theta, -3\cos\theta, 0)$

$\vec{F}(\vec{r}(\theta)) = (-27\sin\theta, 27\cos\theta, 81)$

$\int \vec{F}\cdot d\vec{s} = \int \vec{F}(\vec{r}(\theta))\cdot\vec{v}\,d\theta = \int_0^{2\pi}(-81\sin^2\theta + -81\cos^2\theta)\,d\theta = -\int_0^{2\pi} 81\,d\theta = -162\pi$