Name________________________ ID__________________ Section______

MATH 253	Final Exam	Spring 2004
Sections 504-506	Solutions	P. Yasskin

1-10	/50
11	/ 5
12	/15
13	/30
Total	/100

On the front of the Blue Book, on the Scantron and on this sheet
write your Name, your University ID, your Section and "Final Exam."
On the front of the Blue Book copy the Grading Grid shown at the right.
Enter your Multiple Choice answers on the Scantron
and CIRCLE them on this sheet.

Multiple Choice: (5 points each. No part credit.)

1. Find the area of the parallelogram with vertices $(0,0,0)$, $(4,2,3)$, $(5,4,3)$ and $(1,2,0)$.

a. $\sqrt{8}$

b. 8

c. 3

d. 9 correctchoice

e. 81

$$\vec{u}\times\vec{v} = (4,2,3)\times(1,2,0) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 2 & 3 \\ 1 & 2 & 0 \end{vmatrix} = \hat{\imath}(-6)-\hat{\jmath}(-3)+\hat{k}(6) = (-6,3,6)$$

$$|\vec{u}\times\vec{v}| = \sqrt{36+9+36} = \sqrt{81} = 9$$

2. Find the equation of the line perpendicular to the hyperboloid $xyz = 6$ at the point $(1,2,3)$.

a. $6x+3y+2z = 18$

b. $x+2y+3z = 14$

c. $(x,y,z) = (1+6t, 2-3t, 3+2t)$

d. $(x,y,z) = (6+t, 3+2t, 2+3t)$

e. $(x,y,z) = (1+6t, 2+3t, 3+2t)$ correctchoice

$$P = (1,2,3) \qquad F = xyz \qquad \vec{\nabla}F = (yz,xz,xy) \qquad \vec{N} = \vec{\nabla}F\Big|_{(1,2,3)} = (6,3,2)$$

$$X = P + t\vec{N} = (1,2,3)+t(6,3,2) = (1+6t, 2+3t, 3+2t)$$

3. Find $\left.\frac{\partial z}{\partial v}\right|_{(3,4)}$, given that $z = x^2y^2 - x^3y$ where $x = x(u,v)$ and $y = y(u,v)$ satisfy

$$x(3,4) = 1 \qquad \left.\frac{\partial x}{\partial u}\right|_{(3,4)} = 8 \qquad \left.\frac{\partial x}{\partial v}\right|_{(3,4)} = 6$$

$$y(3,4) = 2 \qquad \left.\frac{\partial y}{\partial u}\right|_{(3,4)} = 7 \qquad \left.\frac{\partial y}{\partial v}\right|_{(3,4)} = 5$$

a. -3

b. 3

c. 27 correctchoice

d. -27

e. 153

$$\left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2xy^2 - 3x^2y|_{(1,2)} = 8 - 6 = 2 \qquad \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 2x^2y - x^3|_{(1,2)} = 4 - 1 = 3$$

$$\left.\frac{\partial z}{\partial v}\right|_{(3,4)} = \left.\frac{\partial z}{\partial x}\right|_{(1,2)} \left.\frac{\partial x}{\partial v}\right|_{(3,4)} + \left.\frac{\partial z}{\partial y}\right|_{(1,2)} \left.\frac{\partial y}{\partial v}\right|_{(3,4)} = 2 \cdot 6 + 3 \cdot 5 = 27$$

4. At $t = 0$ a biker is at $\vec{r} = (2,5)$ and has velocity $\vec{v} = (2,1)$ where distances are measured in meters and time is measured in seconds. The biker measures that at $t = 0$ the temperature is $T(2,5) = 60°$ and the gradient of the temperature is $\vec{\nabla}T(2,5) = (3,4)°$/meter. What is the rate of change of the temperature at $t = 0$ seen by the biker?
HINT: $T(x,y)$ gives the temperature as a function of position and $\vec{r}(t) = (x(t),y(t))$ gives the position of the bike as a function of time. So the composition $T(\vec{r}(t))$ gives the temperature as a function of time.

a. 10 correctchoice

b. 5

c. $\sqrt{5}$

d. 300

e. 12

$$\frac{dT(\vec{r}(t))}{dt} = \vec{\nabla}T \cdot \frac{d\vec{r}}{dt} = (3,4) \cdot (2,1) = 10$$

5. The function $f(x,y) = x^2y^2 - 8x - 4y$ has a critical point at $(x,y) = (1,2)$ which, according to the Second Derivative Test, is

a. a local minimum.

b. a local maximum.

c. a saddle point. correctchoice

d. an inflection point.

e. The Second Derivative Test FAILS.

$f_x = 2xy^2 - 8 \qquad f_y = 2x^2y - 4$

$f_{xx} = 2y^2 \qquad f_{yy} = 2x^2 \qquad f_{xy} = 4xy \qquad D = f_{xx}f_{yy} - f_{xy}{}^2$

$f_{xx}(1,2) = 8 \qquad f_{yy}(1,2) = 2 \qquad f_{xy}(1,2) = 8 \qquad D = 8 \cdot 2 - 8^2 = -48$ saddle

6. Compute $\iint x^2\,dA$ over the region between the parabolas $y = 4 - x^2$ and $y = 8 - 2x^2$.

a. $\frac{32}{15}$

b. $\frac{64}{15}$

c. $\frac{128}{15}$ correctchoice

d. $\frac{256}{15}$

e. $-\frac{256}{15}$

Find where the parabolas intersect: $4 - x^2 = 8 - 2x^2 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

$$\iint x^2\,dA = \int_{-2}^{2}\int_{4-x^2}^{8-2x^2} x^2\,dy\,dx = \int_{-2}^{2}\Big[x^2y\Big]_{y=4-x^2}^{8-2x^2}dx = \int_{-2}^{2} x^2[(8-2x^2)-(4-x^2)]\,dx = \int_{-2}^{2} x^2(4-x^2)\,dx$$

$$= \int_{-2}^{2}(4x^2 - x^4)\,dx = \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = 2\left[\frac{32}{3} - \frac{32}{5}\right] = 64\frac{5-3}{15} = \frac{128}{15}$$

7. Find the average temperature $\bar{T} = \dfrac{\iint T\,dA}{\iint dA}$ of a frying pan which is 4 inches in radius if it is hottest in the center and the temperature decreases toward the edge according to $T = 133° - 3r$.

a. $58.5°$

b. $125°$ correctchoice

c. $127°$

d. $141°$

e. $2000\pi°$

$\iint dA = \int_0^{2\pi}\int_0^4 r\,dr\,d\theta = 2\pi\left[\frac{r^2}{2}\right]_0^4 = 16\pi$ OR $A = \pi R^2 = \pi 4^2 = 16\pi$

$$\iint T\,dA = \int_0^{2\pi}\int_0^4 (133 - 3r)r\,dr\,d\theta = 2\pi\left[133\frac{r^2}{2} - r^3\right]_0^4 = 2\pi(133 \cdot 8 - 64) = 2\pi(1064 - 64) = 2000\pi$$

$$\bar{T} = \frac{2000\pi}{16\pi} = 125°$$

8. Compute $\int_{(1,\ln 2,1)}^{(e,\ln 3,2)} \vec{F} \cdot d\vec{s}$ where $\vec{F} = (yz, xz, xy)$ along the curve $\vec{r}(t) = (e^{t-1}, \ln(1+t), t)$.
HINT: Find a scalar potential for $\vec{F}$ and use the Fundamental Theorem of Calculus for Curves.

a. $2e\ln 3$

b. $2e\ln 3 - \ln 2$ correctchoice

c. $2e\ln 2 - \ln 3$

d. $\ln 2$

e. $\ln 2 - 2e\ln 3$

$\vec{F} = (yz, xz, xy) = \vec{\nabla} f$ for $f = xyz$. So

$$\int_{(1,\ln 2,1)}^{(e,\ln 3,2)} \vec{F} \cdot d\vec{s} = \int_{(1,\ln 2,1)}^{(e,\ln 3,2)} \vec{\nabla} f \cdot d\vec{s} = f(e,\ln 3,2) - f(1,\ln 2,1) = 2e\ln 3 - \ln 2$$

9. Use Green's Theorem to compute $\oint (xy^2 + e^{\sqrt{x}})\,dx + (3x^2y + \cos(y^3))\,dy$ counterclockwise around the boundary of the region between the parabolas $y = x^2$ and $x = y^2$.

a. $\frac{-2}{3}$

b. $\frac{-1}{3}$

c. 0

d. $\frac{1}{3}$ correctchoice

e. $\frac{2}{3}$

By Green's Theorem,

$$\oint (xy^2 + e^{\sqrt{x}})\,dx + (3x^2y + \cos(y^3))\,dy = \iint \left(\frac{\partial}{\partial x}(3x^2y + \cos(y^3)) - \frac{\partial}{\partial y}(xy^2 + e^{\sqrt{x}}) \right) dx\,dy$$

$$= \iint (6xy - 2xy)\,dx\,dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 4xy\,dy\,dx = \int_0^1 \Big[2xy^2 \Big]_{y=x^2}^{\sqrt{x}} dx = \int_0^1 (2x^2 - 2x^5)\,dx$$

$$= \left[\frac{2x^3}{3} - \frac{x^6}{3} \right]_{x=0}^{1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

10. Use Gauss' Theorem to compute $\iint\limits_{\partial C} \vec{F} \cdot d\vec{S}$ for $\vec{F} = (x^3z, y^3z, x^2 + y^2)$ over the total surface of the cylinder $x^2 + y^2 \le 4$ for $0 \le z \le 3$.

a. 36π

b. 48π

c. 72π

d. 96π

e. 108π correctchoice

By Gauss' Theorem, $\iint\limits_{\partial C} \vec{F} \cdot d\vec{S} = \iiint\limits_{C} \vec{\nabla} \cdot \vec{F}\,dV$. Use cylindrical coordinates.

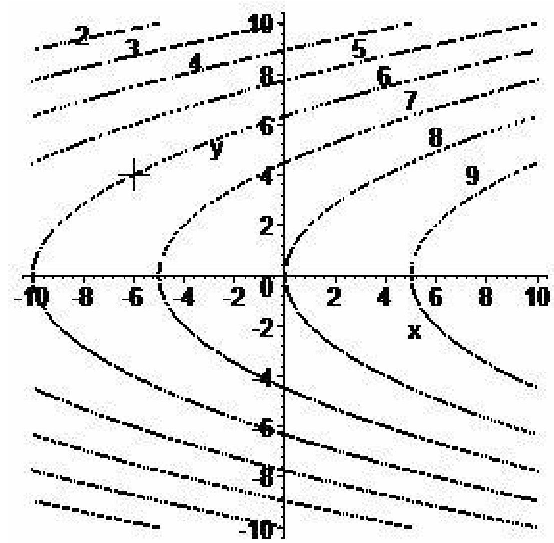
$$\vec{\nabla} \cdot \vec{F} = 3x^2z + 3y^2z = 3r^2z$$

$$\iiint\limits_{C} \vec{\nabla} \cdot \vec{F}\,dV = \int_0^{2\pi} \int_0^2 \int_0^3 3r^2z\,r\,dz\,dr\,d\theta = 6\pi \left[\frac{r^4}{4} \right]_0^2 \left[\frac{z^2}{2} \right]_0^3 = 6\pi(4)\left(\frac{9}{2} \right) = 108\pi$$

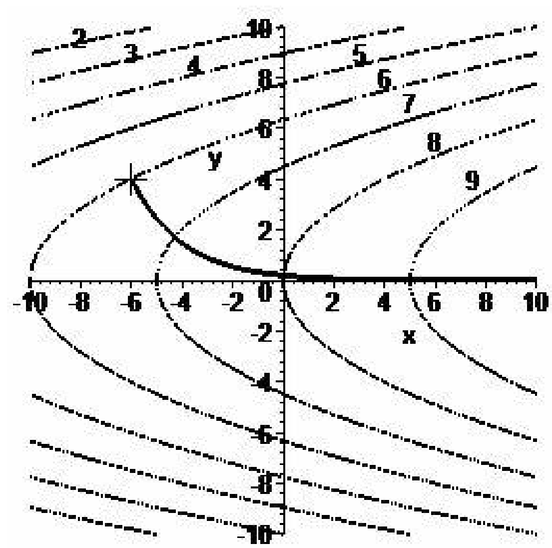
Work Out: (15 points each. Part credit possible.)

Start each problem on a new page of the Blue Book. Number the problem. Show all work.

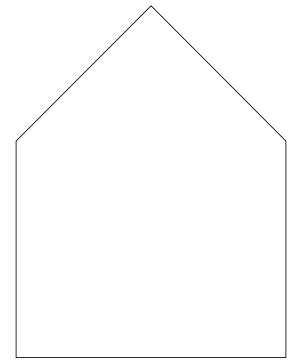
11. (5 points) At the right is the contour plot of a function $f(x,y)$. The contours are labeled by the function values. If you start at the cross at $(-6,4)$ and move so that your velocity is always in the direction of $\vec{\nabla} f,$ the gradient of $f,$ roughly sketch your path on the plot.



You are to draw a curve which starts at the cross, comes down and curves to the right, always perpendicular to each contour it crosses.



12. (15 points) A church window will have the shape of a rectangle with an isosceles right triangle on top. The area of the window is to be $A = 1 + 2\sqrt{2}$. Since the window frame will be gold plated, we want to minimize the perimeter of the window. What are the dimensions of the rectangular part of the window which minimize the perimeter of the whole window?



You MUST use the method of **Lagrange multipliers**.

HINT: If a is the height of the rectangle and b is its width, then $\dfrac{b}{\sqrt{2}}$ is the length of each slanted side of the triangular top.

Let a be the height and b be the width of the rectangle.

The area is constrained to be $A = ab + \dfrac{b^2}{4} = 1 + 2\sqrt{2}$

and we wish to minimize the perimeter $P = 2a + b + 2b/\sqrt{2} = 2a + b + \sqrt{2}\,b$.

$\vec{\nabla}P = \left(2, 1 + \sqrt{2}\right) \qquad \vec{\nabla}A = \left(b, a + \dfrac{b}{2}\right)$

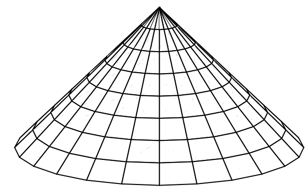
The Lagrange equations are: $\vec{\nabla}P = \lambda\vec{\nabla}A \quad \Rightarrow \quad 2 = \lambda b$ and $1 + \sqrt{2} = \lambda\left(a + \dfrac{b}{2}\right)$

$$\lambda = \frac{2}{b} = \frac{1+\sqrt{2}}{a + \frac{b}{2}} \quad \Rightarrow \quad 2\left(a + \frac{b}{2}\right) = \left(1 + \sqrt{2}\right)b \quad \Rightarrow \quad 2a + b = b + \sqrt{2}\,b \quad \Rightarrow \quad a = \frac{b}{\sqrt{2}}$$

Substitute into the constraint: $\left(\dfrac{b}{\sqrt{2}}\right)b + \dfrac{b^2}{4} = 1 + 2\sqrt{2} \quad \Rightarrow \quad \dfrac{2\sqrt{2}\,b^2}{4} + \dfrac{b^2}{4} = 1 + 2\sqrt{2}$

$$\Rightarrow \quad \frac{b^2}{4}\left(2\sqrt{2} + 1\right) = 1 + 2\sqrt{2} \quad \Rightarrow \quad b^2 = 4 \quad \Rightarrow \quad b = 2 \quad \Rightarrow \quad a = \frac{b}{\sqrt{2}} = \sqrt{2}$$

13. (30 points) Verify Stokes' Theorem $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$



for the vector field $\vec{F} = (-y, x, x^2 + y^2)$ and

the cone $z = 3 - \sqrt{x^2 + y^2} \ge 0$ oriented up and out.

Use the following steps:

a. Parametrize the boundary curve and compute the line integral.

Successively find: $\vec{r}(\theta)$, $\vec{v}(\theta)$, $\vec{F}(\vec{r}(\theta))$, $\oint_{\partial C} \vec{F} \cdot d\vec{s}$

$$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 0)$$

$$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$$

$$\vec{F}(\vec{r}(\theta)) = (-3\sin\theta, 3\cos\theta, 9)$$

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 9\sin^2\theta + 9\cos^2\theta\, d\theta = \int_0^{2\pi} 9\, d\theta = 18\pi$$

b. The cone may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 3 - r)$. Compute the surface integral:

Successively find: $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$, $\vec{\nabla} \times \vec{F}$, $\vec{\nabla} \times \vec{F}\left(\vec{R}(r,\theta)\right)$, $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r & = & (\ \cos\theta, & \sin\theta, & -1) \\ \vec{e}_\theta & = & (-r\sin\theta, & r\cos\theta, & 0\ ) \end{array}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(r\cos\theta) - \hat{\jmath}(-r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (r\cos\theta, r\sin\theta, r)$$

$\vec{N}$ has the correct orientation.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y, & x, & x^2 + y^2 \end{vmatrix} = \hat{\imath}(2y) - \hat{\jmath}(2x) + \hat{k}(1 + 1) = (2y, -2x, 2)$$

$$\vec{\nabla} \times \vec{F}\left(\vec{R}(r,\theta)\right) = (2r\sin\theta, -2r\cos\theta, 2)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = 2r^2\sin\theta\cos\theta - 2r^2\cos\theta\sin\theta + 2r = 2r$$

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_C \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^3 2r\, dr\, d\theta = 2\pi[r^2]_{r=0}^3 = 18\pi$$