Name________________________ ID__________________

1-8	/40	11	/15
9	/ 5	12	/15
10	/15	13	/15
Total			/105

MATH 253 Exam 2 Spring 2007

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the equation of the line perpendicular to the graph of

$xyz - x^2 - y^2 - z^2 = -8$ at the point $(1,2,3)$.

Where does this line intersect the xz-plane?

a. $(-7,0,-5)$

b. $(-7,0,11)$

c. $(9,0,-5)$ Correct Choice

d. $(9,0,11)$

e. $(5,0,-1)$

Let $F = xyz - x^2 - y^2 - z^2$. Then $\vec{\nabla}F = \langle yz - 2x, xz - 2y, xy - 2z\rangle$.

Then the normal at $P = (1,2,3)$ is $\vec{N} = \vec{\nabla}F\big|_P = \langle 4,-1,-4\rangle$.

This is also the tangent vector to the perpendicular line. So $\vec{v} = \langle 4,-1,-4\rangle$.

So the perpendicular line is $X = P + t\vec{v} = (1,2,3) + t\langle 4,-1,-4\rangle$ or

$x = 1 + 4t \qquad y = 2 - t \qquad z = 3 - 4t$

This intersects the xz-plane when $y = 0$ or $t = 2$.

So $x = 1 + 4t = 9$ and $z = 3 - 4t = -5$.

2. The point $(x,y) = (1,2)$ is a critical point of the function $f(x,y) = (x^2 + y^2 - 4)^2 - 4x - 8y$.
Use the Second Derivative Test to classify it as a

a. local maximum

b. local minimum Correct Choice

c. inflection point

d. saddle point

e. Test Fails

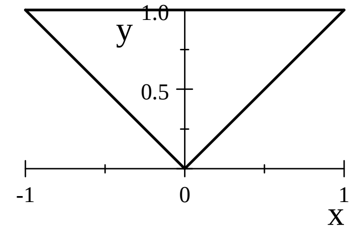
$f_x = 2(x^2 + y^2 - 4)2x - 4 \qquad f_y = 2(x^2 + y^2 - 4)2y - 8 \qquad f_x(1,2) = f_y(1,2) = 0$

$f_{xx} = 2(2x)2x + 2(x^2 + y^2 - 4)2 \qquad f_{yy} = 2(2y)2y + 2(x^2 + y^2 - 4)2 \qquad f_{xy} = 2(2y)2x$

$f_{xx}(1,2) = 8 + 4 = 12 > 0 \qquad f_{yy}(1,2) = 32 + 4 = 36 > 0 \qquad f_{xy}(1,2) = 16$

$D = f_{xx}f_{yy} - f_{xy}^{\ 2} = 12 \cdot 36 - 16^2 = 176 > 0$ local minimum

3. Find the center of mass of the triangle with vertices $(0,0)$, $(1,1)$ and $(-1,1)$ if the mass density is $\rho = y$.



a. $\left(0, \frac{1}{3}\right)$

b. $\left(0, \frac{1}{2}\right)$

c. $\left(0, \frac{2}{3}\right)$

d. $\left(0, \frac{3}{4}\right)$ Correct Choice

e. $\left(0, \frac{4}{5}\right)$

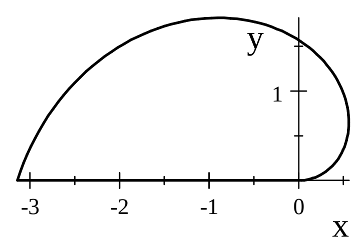
$\bar{x} = 0$ by symmetry, Use a y-integral. $0 \le y \le 1$ $x = -y$ $x = y$

$$M = \iint \rho\, dA = \int_0^1 \int_{-y}^{y} y\, dx\, dy = \int_0^1 \Big[yx \Big]_{x=-y}^{y} dy = 2\int_0^1 y^2 dy = 2\left[\frac{y^3}{3} \right]_{y=0}^{1} = \frac{2}{3}$$

$$y\text{-mom} = M_x = \iint y\rho\, dA = \int_0^1 \int_{-y}^{y} y^2\, dx\, dy = \int_0^1 \Big[y^2 x \Big]_{x=-y}^{y} dy = 2\int_0^1 y^3 dy = 2\left[\frac{y^4}{4} \right]_{y=0}^{1} = \frac{1}{2}$$

$$\bar{y} = \frac{y\text{-mom}}{M} = \frac{M_x}{M} = \frac{1}{2}\,\frac{3}{2} = \frac{3}{4}$$

4. Compute $\iint (x^2 + y^2)\, dA$ over the region bounded by the polar curve $r = \theta$ and the x-axis.



a. $\frac{\pi^5}{20}$ Correct Choice

b. $\frac{\pi^4}{16}$

c. $\frac{\pi^4}{12}$

d. $\frac{\pi^3}{9}$

e. $\frac{\pi^3}{6}$

$$\int_0^{\pi} \int_0^{\theta} r^2\, r\, dr\, d\theta = \int_0^{\pi} \left[\frac{r^4}{4} \right]_0^{\theta} d\theta = \frac{1}{4} \int_0^{\pi} \theta^4\, d\theta = \frac{1}{4} \left[\frac{\theta^5}{5} \right]_0^{\pi} = \frac{\pi^5}{20}$$

5. Find the mass of the $1/8$ of the solid sphere $x^2+y^2+z^2 \le 16$ in the first octant if the mass density is $\delta = z$.

a. 64π

b. 16π Correct Choice

c. 8π

d. 4π

e. π

$$M = \iiint \delta\, dV = \iiint z\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho(\cos\varphi)\,\rho^2(\sin\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos\varphi\sin\varphi\,d\varphi \int_0^4 \rho^3\,d\rho = \frac{\pi}{2}\left[\frac{\sin^2\varphi}{2}\right]_{\varphi=0}^{\pi/2}\left[\frac{\rho^4}{4}\right]_{\rho=0}^{4} = \frac{\pi}{2}\frac{1}{2}4^3 = 16\pi$$

6. Find the volume of the solid between the cone $z = 2\sqrt{x^2+y^2}$ and the paraboloid $z = 8 - x^2 - y^2$.
HINT: Find the radius where the cone and paraboloid intersect.

a. 30π

b. $\frac{80\pi}{3}$

c. $\frac{40\pi}{3}$ Correct Choice

d. $\frac{20\pi}{3}$

e. $\frac{10\pi}{3}$

$z = 2r = 8 - r^2 \qquad r^2 + 2r - 8 = 0 \qquad (r-2)(r+4) = 0 \qquad r = 2$ since $r \ge 0$.

$$V = \iiint 1\, dV = \int_0^{2\pi}\int_0^2\int_{2r}^{8-r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^2\Big[rz\Big]_{z=2r}^{8-r^2} dr = 2\pi\int_0^2 r(8 - r^2 - 2r)\,dr$$

$$= 2\pi\int_0^2 (8r - r^3 - 2r^2)\,dr = 2\pi\left[4r^2 - \frac{r^4}{4} - \frac{2r^3}{3}\right]_{r=0}^{2} = 2\pi\left(16 - 4 - \frac{16}{3}\right) = \frac{40\pi}{3}$$

7. Compute $\iint_S \vec{F}\cdot d\vec{S}$ for the vector field $\vec{F} = (z, z, x+y)$ over the surface S which is parametrized by $\vec{R}(u,v) = (u+v, u-v, uv)$ for $0 \le u \le 2$ and $0 \le v \le 3$ and oriented along $\vec{N} = \vec{e}_u \times \vec{e}_v$.

a. -60

b. -12

c. 0

d. 12 Correct Choice

e. 60

$$\vec{R}(u,v) = (u+v, u-v, uv) \qquad \begin{matrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (\ 1, & 1, & v\ ) \\ \vec{e}_v = & (\ 1, & -1, & u\ ) \end{matrix}$$

$$\vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(u+v) - \hat{\jmath}(u-v) + \hat{k}(-1-1) = (u+v, v-u, -2)$$

$$\vec{F} = (z, z, x+y) \qquad \vec{F}\big(\vec{R}(u,v)\big) = (uv, uv, 2u) \qquad \vec{F}\cdot\vec{N} = uv(u+v) + uv(v-u) + 2u(-2) = u(2v^2 - 4)$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint \vec{F}\big(\vec{R}(u,v)\big)\cdot\vec{N}\,du\,dv = \int_0^3\int_0^2 u(2v^2-4)\,du\,dv = \left[\frac{u^2}{2}\right]_0^2\left[\frac{2v^3}{3} - 4v\right]_0^3 = (2)(18-12) = 12$$

8. Compute $\iint_C \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (xz, yz, z^2)$ over the cylindrical surface $x^2 + y^2 = 9$ for $0 \le z \le 2$ oriented outward.

a. 18π

b. $\dfrac{62}{3}\pi$

c. 21π

d. $\dfrac{124}{3}\pi$

e. 36π Correct Choice

$$\vec{R}(\theta,z) = (3\cos\theta, 3\sin\theta, z) \qquad \begin{matrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta = & (-3\sin\theta, & 3\cos\theta, & 0) \\ \vec{e}_z = & (\quad 0, & 0, & 1) \end{matrix}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(3\cos\theta) - \hat{\jmath}(-3\sin\theta) + \hat{k}(0) = (3\cos\theta, 3\sin\theta, 0)$$

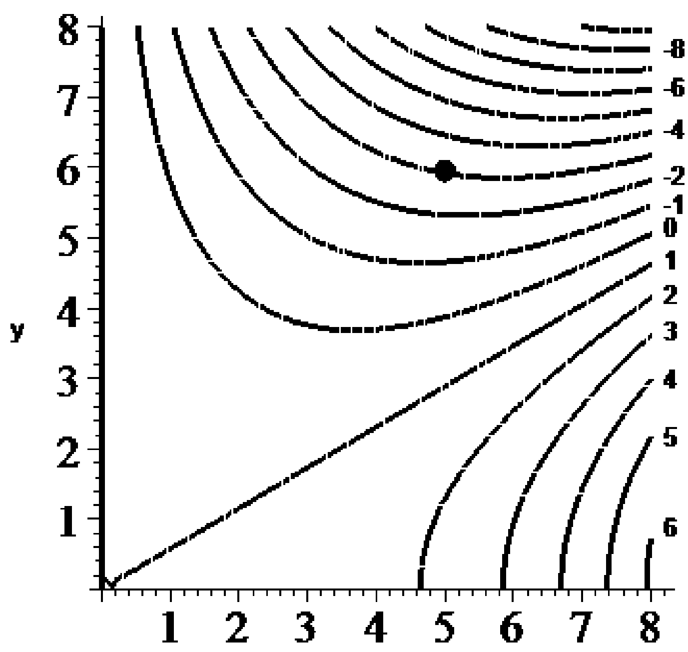
$$\vec{F} = (xz, yz, z^2) \qquad \vec{F}\left(\vec{R}(\theta,z)\right) = (3z\cos\theta, 3z\sin\theta, z^2) \qquad \vec{F} \cdot \vec{N} = 9z\cos^2\theta + 9z\sin^2\theta = 9z$$

$$\iint_C \vec{F} \cdot d\vec{S} = \int\int \vec{F}\left(\vec{R}(\theta,z)\right) \cdot \vec{N}\, d\theta\, dz = \int_0^2 \int_0^{2\pi} 9z\, d\theta\, dz = 9(2\pi)\left[\frac{z^2}{2}\right]_0^2 = 36\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (5 points) At the right is the contour plot of a function $f(x,y)$. If you **start** at the dot at $(5,6)$ and move so that your velocity is always in the direction of $\vec{\nabla} f$, the gradient of f, roughly sketch your path on the plot.

NOTE : The numbers on the right are the values of f on each level curve.



The curve starts at $(5,6)$ goes down and curves to the right towards higher values of the function f, always perpendicular to each level curve. It should not go up.

10. (15 points) An aquarium in the shape of a rectangular solid has a base made of marble which costs 6 cents per square inch, a back and sides made of mirrored glass which costs 2 cents per square inch and a front made of clear glass which costs 1 cent per square inch. There is no top. If the volume of the aquarium is 9000 cubic inches, what are the dimensions of the cheapest such aquarium?

Let x be the length side to side, y be the width front to back, and z be the height.

The cost is $C = 6xy + 2(xz + 2yz) + 1xz = 6xy + 3xz + 4yz$

The volume constraint is $V = xyz = 9000$ which we solve for $z = \dfrac{9000}{xy}$

So the cost becomes $C = 6xy + \dfrac{27000}{y} + \dfrac{36000}{x}$

We want to minimize C. So we set the partials equal to zero:

$C_x = 6y - \dfrac{36000}{x^2} = 0 \qquad C_y = 6x - \dfrac{27000}{y^2} = 0$

$y = \dfrac{6000}{x^2} \qquad x = \dfrac{27000}{6y^2} = \dfrac{4500}{\left(\dfrac{6000}{x^2}\right)^2} = \dfrac{4500x^4}{6000 \cdot 6000} = \dfrac{1}{8000}x^4$

Cancel an x and solve for x then y and z:

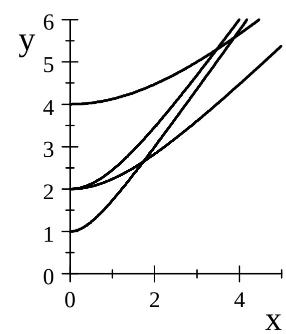
$1 = \dfrac{x^3}{8000}, \quad x = 20 \quad y = \dfrac{6000}{x^2} = \dfrac{6000}{400} = 15 \quad z = \dfrac{9000}{xy} = \dfrac{9000}{300} = 30$

So the dimensions are: $\boxed{x = 20 \quad y = 15 \quad z = 30}$

11. (15 points) Compute $\iint xy\,dA$ over the "diamond" shaped region bounded by the curves

$$y^2 - x^2 = 4 \qquad y^2 - x^2 = 16$$
$$y^2 - 2x^2 = 1 \qquad y^2 - 2x^2 = 4$$

HINT: Let $u = y^2 - x^2$ and $v = y^2 - 2x^2$.



$u - v = y^2 - x^2 - y^2 + 2x^2 = x^2 \qquad 2u - v = 2y^2 - 2x^2 - y^2 + 2x^2 = y^2 \qquad x = \sqrt{u-v} \qquad y = \sqrt{2u-v}$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{matrix}\right\| = \left\|\begin{matrix} \dfrac{1}{2\sqrt{u-v}} & \dfrac{2}{2\sqrt{2u-v}} \\ \dfrac{-1}{2\sqrt{u-v}} & \dfrac{-1}{2\sqrt{2u-v}} \end{matrix}\right\|$$

$$= \left| \frac{1}{2\sqrt{u-v}}\frac{-1}{2\sqrt{2u-v}} - \frac{2}{2\sqrt{2u-v}}\frac{-1}{2\sqrt{u-v}} \right| = \frac{1}{4\sqrt{u-v}\sqrt{2u-v}}$$

$xy = \sqrt{u-v}\sqrt{2u-v} \qquad 4 \le u \le 16 \qquad 1 \le v \le 4$

$$\iint xy\,dA = \int_1^4 \int_4^{16} \sqrt{u-v}\sqrt{2u-v}\,\frac{1}{4\sqrt{u-v}\sqrt{2u-v}}\,du\,dv = \int_1^4 \int_4^{16} \frac{1}{4}\,du\,dv = \frac{1}{4}(16-4)(4-1) = 9$$

12. (15 points) Find the average temperature on the hemisphere surface $x^2 + y^2 + z^2 = 9$, with $z \ge 0$, if the temperature is $T = z$.

NOTE : The average of a function f is $f_{\text{ave}} = \dfrac{\iint f\,dS}{\iint dS}$. HINT: Parametrize the hemisphere.

$$\vec{R}(\varphi,\theta) = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi)$$

$$\begin{array}{lccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\varphi = & (3\cos\varphi\cos\theta, & 3\cos\varphi\sin\theta, & -3\sin\varphi) \\ \vec{e}_\theta = & (-3\sin\varphi\sin\theta, & 3\sin\varphi\cos\theta, & 0 \quad) \end{array}$$

$$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = \hat{\imath}(9\sin^2\varphi\cos\theta) - \hat{\jmath}(-9\sin^2\varphi\sin\theta) + \hat{k}(9\sin\varphi\cos\varphi\cos^2\theta + 9\sin\varphi\cos\varphi\sin^2\theta)$$

$$= (9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi)$$

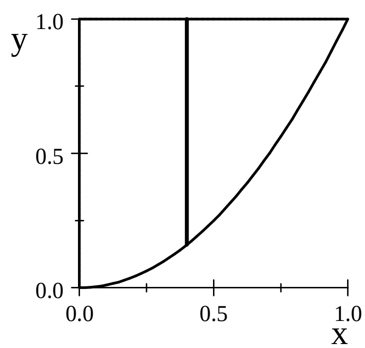
$$\left|\vec{N}\right| = \sqrt{81\sin^4\varphi\cos^2\theta + 81\sin^4\varphi\sin^2\theta + 81\sin^2\varphi\cos^2\varphi} = 9\sin\varphi$$

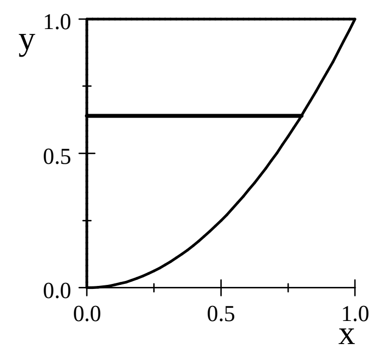
$$\iint dS = \int_0^{2\pi}\int_0^{\pi/2}\left|\vec{N}\right| d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} 9\sin\varphi\,d\varphi\,d\theta = 18\pi \qquad (\text{Area of hemisphere} = \tfrac{1}{2}(4\pi R^2).)$$

$$\iint T\,dS = \int_0^{2\pi}\int_0^{\pi/2} z\left|\vec{N}\right| d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} 3\cos\varphi\, 9\sin\varphi\,d\varphi\,d\theta = 27 \cdot 2\pi\left[\frac{\sin^2\varphi}{2}\right]_{\varphi=0}^{\pi/2} = 27\pi$$

$$T_{\text{ave}} = \frac{\iint T\,dS}{\iint dS} = \frac{27\pi}{18\pi} = \frac{3}{2}$$

13. (15 points) Sketch the region of integration and then compute the integral $\int_0^1\int_{x^2}^1 x^3\cos(y^3)\,dy\,dx$.





Reverse the order of integration:

$$\int_0^1\int_{x^2}^1 x^3\cos(y^3)\,dy\,dx = \int_0^1\int_0^{\sqrt{y}} x^3\cos(y^3)\,dx\,dy = \int_0^1\left[\frac{x^4}{4}\cos(y^3)\right]_{x=0}^{\sqrt{y}} dy = \int_0^1 \frac{y^2}{4}\cos(y^3)\,dy$$

Substitute: $u = y^3$ $\quad du = 3y^2\,dy$ $\quad y^2\,dy = \frac{1}{3}du$

$$\int_0^1\int_{x^2}^1 x^3\cos(y^3)\,dy\,dx = \frac{1}{12}\int\cos(u)\,du = \frac{1}{12}\sin(u) = \frac{1}{12}\sin(y^3)\Big|_0^1 = \frac{1}{12}\sin 1$$