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1-9	/54	12	/20
10	/15	13	/ 6
11	/15		
Total			/110

MATH 253 Final Exam Spring 2007

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Consider the triangle with vertices $A = (1,-1,2)$, $B = (2,3,1)$ and $C = (4,2,2)$.
Which vector is perpendicular to the plane of the triangle?

a. $(1,1,3)$

b. $(-1,-1,-3)$

c. $(-1,1,-3)$

d. $(1,1,-3)$

e. $(1,-1,-3)$ Correct Choice

$\overrightarrow{AB} = B - A = (1,4,-1)$ $\quad \overrightarrow{AC} = C - A = (3,3,0)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 4 & -1 \\ 3 & 3 & 0 \end{vmatrix} = \hat{\imath}(3) - \hat{\jmath}(3) + \hat{k}(3-12) = (3,-3,-9)$$ is perpendicular.

$(1,-1,-3)$ has the same direction.

2. For the "helix" **curve** $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 3\theta)$ find the unit binormal $\hat{B} = \dfrac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|}$.

a. $(12\sin\theta, 12\cos\theta, 16)$

b. $(-12\sin\theta, -12\cos\theta, -16)$

c. $\left(\frac{3}{5}\sin\theta, \frac{3}{5}\cos\theta, \frac{4}{5}\right)$

d. $\left(-\frac{3}{5}\sin\theta, -\frac{3}{5}\cos\theta, -\frac{4}{5}\right)$

e. $\left(\frac{3}{5}\sin\theta, -\frac{3}{5}\cos\theta, \frac{4}{5}\right)$ Correct Choice

$\vec{v} = (-4\sin\theta, 4\cos\theta, 3)$ $\quad \vec{a} = (-4\cos\theta, -4\sin\theta, 0)$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4\sin\theta & 4\cos\theta & 3 \\ -4\cos\theta & -4\sin\theta & 0 \end{vmatrix} = \hat{\imath}(12\sin\theta) - \hat{\jmath}(12\cos\theta) + \hat{k}(16\sin^2\theta + 16\cos^2\theta) = (12\sin\theta, -12\cos\theta, 16)$$

$|\vec{v}\times\vec{a}| = \sqrt{144+256} = 20$

$$\hat{B} = \frac{\vec{v}\cdot\vec{a}}{|\vec{v}\cdot\vec{a}|} = \left(\frac{3}{5}\sin\theta, -\frac{3}{5}\cos\theta, \frac{4}{5}\right)$$

3. Find the equation of the plane tangent to $z = xy^2 + x^3y$ at $(x,y) = (1,2)$.
What is the z-intercept?

a. $(0,0,6)$

b. $(0,0,-6)$

c. $(0,0,14)$

d. $(0,0,-14)$ Correct Choice

e. $(0,0,26)$

$f(x,y) = xy^2 + x^3y \qquad f(1,2) = 6$
$f_x(x,y) = y^2 + 3x^2y \qquad f_x(1,2) = 10$
$f_y(x,y) = 2xy + x^3 \qquad f_y(1,2) = 5$
$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 6 + 10(x-1) + 5(y-2)$
z-intercept: $x = 0 \qquad y = 0 \qquad z = 6 + 10(-1) + 5(-2) = -14$

4. The temperature in a room is given by $T = 72 + xyz$. What is the time rate of change of the temperature as seen by a fly located at $P = (3,2,1)$ with velocity $\vec{v} = (2,2,1)$?

a. 4

b. 11

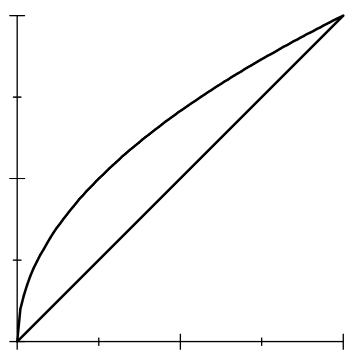
c. 16 Correct Choice

d. 18

e. 22

$\vec{\nabla}T = (yz,xz,xy) \qquad \vec{\nabla}T\big|_P = (2,3,6) \qquad \frac{dT}{dt} = \vec{v}\cdot\vec{\nabla}T\big|_P = 4+6+6 = 16$

5. Find the volume under the surface $z = 2x^2y$ above the region bounded by $y = x$ and $y = 2\sqrt{x}$.
The base is shown at the right.



a. $\frac{256}{5}$ Correct Choice

b. $\frac{320}{3}$

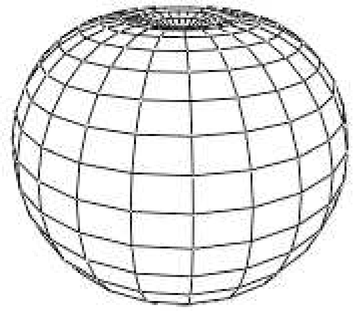
c. $\frac{64}{7}$

d. $\frac{320}{7}$

e. $\frac{64}{5}$

The curves intersect when $x = 2$ or $x^2 = 4x$ or $x = 0,4$
$$V = \int_0^4\int_x^{2\sqrt{x}} 2x^2y\,dy\,dx = \int_0^4 [x^2y^2]_{y=x}^{2\sqrt{x}}\,dx = \int_0^4 (4x^3 - x^4)\,dx = \left[x^4 - \frac{x^5}{5}\right]_{x=0}^4 = \frac{4^4}{5} = \frac{256}{5}$$

6. Find the mass of the solid apple given in spherical coordinates by $\rho = 1 - \cos\phi$ if the volume mass density is $\delta = \rho$.



a. $\frac{2}{5}\pi$

b. $\frac{8}{3}\pi$

c. $\frac{16}{5}\pi$ Correct Choice

d. 8π

e. $\frac{64}{15}\pi$

$$M = \iiint \delta\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\varphi} \rho\,\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi \int_0^{\pi}\left[\frac{\rho^4}{4}\right]_{\rho=0}^{1-\cos\varphi} \sin\varphi\, d\varphi = \frac{\pi}{2}\int_0^{\pi}(1-\cos\varphi)^4 \sin\varphi\, d\varphi$$

$u = 1 - \cos\varphi \qquad du = \sin\theta\, d\theta$

$$M = \frac{\pi}{2}\int_0^2 u^4\, du = \frac{\pi}{2}\left[\frac{u^5}{5}\right]_0^2 = \frac{16}{5}\pi$$

7. Compute $\int_{\vec{r}} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (1+yz, 1+xz, 1+xy)$ along the curve $\vec{r}(t) = (\ln(1+t), t\ln(1+t), t^2\ln(1+t))$ between $t = 0$ and $t = 1$.

HINT: Find a scalar potential and use the Fundamental Theorem of Calculus for Curves.

a. $3\ln 2 + 3(\ln 2)^3$

b. $3\ln 2 + (\ln 2)^3$ Correct Choice

c. $3\ln 2 - 3(\ln 2)^3$

d. $3\ln 2 - (\ln 2)^3$

e. $-3\ln 2 + 3(\ln 2)^3$

$\vec{F} = \vec{\nabla} f$ for $f = x + y + z + xyz$

$A = \vec{r}(0) = (0,0,0) \qquad B = \vec{r}(1) = (\ln 2,\ \ln 2,\ \ln 2)$

$$\int_{\vec{r}} \vec{F} \cdot d\vec{s} = \int_{\vec{r}} \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = 3\ln 2 + (\ln 2)^3$$

8. Compute $\oint_C \vec{F} \cdot d\vec{s}$ for $\vec{F} = (x^2y - y^3, x^3 - xy^2)$ counterclockwise around the circle $x^2 + y^2 = 4$.

HINT: Use Green's Theorem.

a. $\dfrac{16}{3}\pi$

b. 8π

c. $\dfrac{32}{3}\pi$

d. 16π Correct Choice

e. 32π

$P = x^2y - y^3 \qquad Q = x^3 - xy^2 \qquad \dfrac{\partial Q}{\partial x} = 3x^2 - y^2 \qquad \dfrac{\partial P}{\partial y} = x^2 - 3y^2$

$\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2x^2 + 2y^2 = 2r^2$

$$\oint_C \vec{F} \cdot d\vec{s} = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_0^{2\pi}\int_0^2 2r^2\, r\, dr\, d\theta = 2\pi\left[\frac{r^4}{2}\right]_0^2 = 16\pi$$

9. Compute $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the paraboloid $z = 9 - x^2 - y^2$ for $z \ge 0$,

oriented up, for the vector field $\vec{F} = (z + y, z - x, 2z)$.

HINT: Use Stokes' Theorem. Parametrize the boundary.

a. -18π Correct Choice

b. 0

c. 9π

d. 18π

e. 36π

The boundary is the circle $x^2 + y^2 = 9$ for $z = 0$.

It may be parametrized as $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 0)$.

$\vec{v} = (-3\sin\theta, 3\cos\theta, 0)$ Oriented counterclockwise as seen from above.

$\vec{F}(\vec{r}(\theta)) = (3\sin\theta, -3\cos\theta, 0)$

$$\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial P} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -9\, d\theta = -18\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (15 points) Find the point (x,y,z) in the first octant on the surface $z = \frac{27}{x} + \frac{64}{y}$ which is closest to the origin.

Minimize the square of the distance to the origin $f = x^2 + y^2 + z^2$

subject to the constraint that the point lies on the surface $z = \frac{27}{x} + \frac{64}{y}$.

Eliminate a constraint: Minimize $f = x^2 + y^2 + \left(\frac{27}{x} + \frac{64}{y}\right)^2$

$f_x = 2x + 2\left(\frac{27}{x} + \frac{64}{y}\right)\left(\frac{-27}{x^2}\right) = 0 \qquad f_y = 2y + 2\left(\frac{27}{x} + \frac{64}{y}\right)\left(\frac{-64}{y^2}\right) = 0$

Multiply the first equation by $\frac{x^2}{54}$ and the second equation by $\frac{y^2}{128}$:

(1.) $\frac{x^3}{27} = \left(\frac{27}{x} + \frac{64}{y}\right)$ (2.) $\frac{y^3}{64} = \left(\frac{27}{x} + \frac{64}{y}\right)$

Equate these to obtain $\frac{x}{3} = \frac{y}{4}$ and plug back into (1.) to obtain:

$\frac{x^3}{27} = \left(\frac{27}{x} + \frac{16 \cdot 3}{x}\right) = \frac{75}{x}$

Cross multiply: $x^4 = 75 \cdot 27 = 3^4 \cdot 5^2$ So $x = 3\sqrt{5}$ and $y = 4\sqrt{5}$

and $z = \frac{27}{x} + \frac{64}{y} = \frac{27}{3\sqrt{5}} + \frac{64}{4\sqrt{5}} = \frac{25}{\sqrt{5}} = 5\sqrt{5}$.

11. (15 points) Find the mass and z-component of the center of mass of the "twisted cubic" **curve** $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ for $0 \le t \le 1$ if the density is $\rho = 3xz + 3y^2$.

$\vec{v} = (1, 2t, 2t^2) \qquad |\vec{v}| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2 \qquad \rho = 3t\frac{2}{3}t^3 + 3(t^2)^2 = 5t^4$

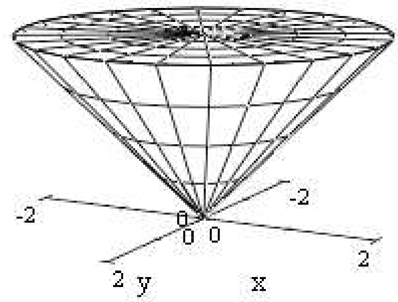
$M = \int \rho\, ds = \int_0^1 \rho |\vec{v}|\, dt = \int_0^1 5t^4(1 + 2t^2)\, dt = 5\int_0^1 (t^4 + 2t^6)\, dt = 5\left[\frac{t^5}{5} + 2\frac{t^7}{7}\right]_0^1 = 5\left(\frac{1}{5} + \frac{2}{7}\right) = \frac{17}{7}$

z-mom $= M_{xy} = \int z\rho\, ds = \int_0^1 z\rho|\vec{v}|\, dt = \frac{2}{3}\int_0^1 t^3 5t^4(1 + 2t^2)\, dt = \frac{10}{3}\int_0^1 (t^7 + 2t^9)\, dt$

$= \frac{10}{3}\left[\frac{t^8}{8} + 2\frac{t^{10}}{10}\right]_0^1 = \frac{10}{3}\left(\frac{1}{8} + \frac{1}{5}\right) = \frac{13}{12}$

$\bar{z} = \frac{z\text{-mom}}{M} = \frac{M_{xy}}{M} = \frac{13}{12}\frac{7}{17} = \frac{91}{204}$

12. (20 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iint_{\partial V} \vec{F}\cdot d\vec{S}$



for the vector field $\vec{F} = (xz, yz, -2z^2)$ and

the volume above the cone C: $z = \sqrt{x^2+y^2}$ for $z \le 2$

and below the disk D: $x^2+y^2 \le 4$ with $z = 2$.

Be sure to check and explain the orientations.

Use the following steps.

a. Compute the divergence $\vec{\nabla}\cdot\vec{F}$ and the volume integral $\iiint_V \vec{\nabla}\cdot\vec{F}\,dV$.

$$\vec{\nabla}\cdot\vec{F} = z + z - 4z = -2z$$

$$\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^2\int_r^2 -2zr\,dz\,dr\,d\theta = -2\pi\int_0^2\Big[z^2\Big]_{z=r}^2 r\,dr = -2\pi\int_0^2 (4-r^2)\,r\,dr$$

$$= -2\pi\left[2r^2 - \frac{r^4}{4}\right]_0^2 = -2\pi(8-4) = -8\pi$$

b. Parametrize the disk, D, and compute the surface integral:
Successively find: $\vec{R}(r,\theta)$, $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$, check orientation, $\vec{F}\big(\vec{R}(r,\theta)\big)$, $\iint_D \vec{F}\cdot d\vec{S}$.

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2)$$

$$\begin{array}{rccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 0) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (0,0,r)$$

$\vec{N}$ has the correct orientation which is up.

$$\vec{F}\big(\vec{R}(r,\theta)\big) = (xz, yz, -2z^2) = (2r\cos\theta, 2r\sin\theta, -8)$$

$$\vec{F}\cdot\vec{N} = -8r$$

$$\iint_D \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^2 -8r\,dr\,d\theta = 2\pi[-4r^2]_0^2 = -32\pi$$

c. The cone, C, may be parametrize as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.
Compute the surface integral:
Successively find: $\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$, check orientation, $\vec{F}\left(\vec{R}(r,\theta)\right)$, $\iint\limits_P \vec{F} \cdot d\vec{S}$.

$$\begin{array}{lccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 1) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$$

$\vec{N}$ needs to be oriented down. So reverse $\vec{N}$:

$$\vec{N} = (r\cos\theta, r\sin\theta, -r)$$

$$\vec{F}\left(\vec{R}(r,\theta)\right) = (xz, yz, -2z^2) = (r^2\cos\theta, r^2\sin\theta, -2r^2)$$

$$\vec{F} \cdot \vec{N} = r^3\cos^2\theta + r^3\sin^2\theta + 2r^3 = 3r^3$$

$$\iint\limits_P \vec{F} \cdot d\vec{S} = \iint\limits_P \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 3r^3\, dr\, d\theta = 2\pi\left[\frac{3r^4}{4}\right]_0^2 = 24\pi$$

d. Combine $\iint\limits_D \vec{F} \cdot d\vec{S}$ and $\iint\limits_P \vec{F} \cdot d\vec{S}$ to get $\iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$.

$$\iint\limits_{\partial V} \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot d\vec{S} + \iint\limits_P \vec{F} \cdot d\vec{S} = -32\pi + 24\pi = -8\pi$$

which agrees with part (a).

13. (6 points) Select the Project that you worked on and then answer the questions in 1 or 2 sentences.

___Gauss' Law and Ampere's Law

One of the electric fields produced a charge density of zero: $\rho_c = \frac{1}{4\pi}\vec{\nabla} \cdot \vec{E} = 0$.

Was there a charge and how did you know? Why was Gauss' Theorem not violated?

___Interpretation of Divergence and Curl

The divergence can be defined using either derivatives or a limit of an integral.

Which Theorem was used to prove their equivalence.

___Skimpy Donut

For the minimal donut, what was the relation between a and b?

For the maximal donut, what was the value of b?

___Volume Between a Surface and Its Tangent Plane

When minimizing over a square, which tangent point (a,b) minimizes the volume?

___Hypervolume of a Hypersphere

The volume enclosed by a sphere of radius R in $\mathbb{R}^n$ is $V_n = k\pi^p R^q$.

What are the values of p and q when $n = 4$? What are the values of p and q when $n = 5$?

___Average Temperatures

What was the shape of the probe used to measure the temperature of the water in the pot?

Which Maple command did you use when Maple was unable to compute the integrals?

___Center of Mass of Planet X

In computing the mass of the water, how did you ensure you only integrated where the land level was below sea level.

Which Maple command did you use when Maple was unable to compute the integrals?