Name______________________________ Sec__________

MATH 251/253 Exam 2 Spring 2008

Sections 508/200,501,502 Solutions P. Yasskin

1-12	/60	14	/15
13	/15	15	/15
Total			/105

Multiple Choice: (5 points each. No part credit.)

1. Find the directional derivative of $f = xyz$ at the point $(x, y, z) = (1, 2, 3)$ **in the direction of** the vector $\vec{v} = (3, 4, 12)$.

a. $\frac{47}{13}$

b. 30

c. $\frac{30}{13}$

d. 54

e. $\frac{54}{13}$ Correct Choice

$\vec{\nabla} f = (yz, xz, xy) \qquad \vec{\nabla} f\Big|_{(1,2,3)} = (6, 3, 2) \qquad \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{13}(3, 4, 12)$

$D_{\hat{v}} f = \hat{v} \cdot \vec{\nabla} f\Big|_{(1,2,3)} = \frac{1}{13}(3, 4, 12) \cdot (6, 3, 2) = \frac{54}{13}$

2. The point $(2, 1)$ is a critical point of the function $f = (x^3 - 3x^2)(y^3 - 3y)$. Use the 2nd Derivative Test to classify $(2, 1)$.

a. local minimum

b. local maximum Correct Choice

c. saddle point

d. inflection point

e. The test fails.

$f_x = (3x^2 - 6x)(y^3 - 3y) \qquad f_y = (x^3 - 3x^2)(3y^2 - 3)$

$f_{xx} = (6x - 6)(y^3 - 3y) \qquad f_{yy} = (x^3 - 3x^2)6y \qquad f_{xy} = (3x^2 - 6x)(3y^2 - 3)$

$f_{xx}(2, 1) = (6)(-2) = -12 < 0 \qquad f_{yy}(2, 1) = (-4)6 = -24 < 0 \qquad f_{xy}(2, 1) = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 288 > 0$ local maximum

3. Compute $\int_1^2 \int_{1/y}^1 y e^{xy}\, dx\, dy$

a. $e^2 - 2e$ Correct Choice

b. $e^2 - e$

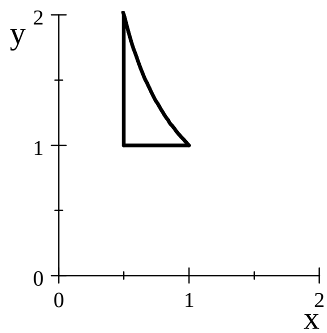
c. $e^2 - 2e - 1$

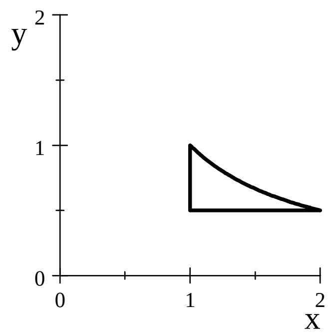
d. $e^2 - e - 1$

e. $e^2 - 2$

$$\int_1^2 \int_{1/y}^1 y e^{xy}\, dx\, dy = \int_1^2 \left[e^{xy} \right]_{x=1/y}^1 dy = \int_1^2 [e^y - e]\, dy = \left[e^y - ey \right]_{y=1}^2 = e^2 - 2e$$

4. The region of integration of the integral in the previous problem is:

a.


b.


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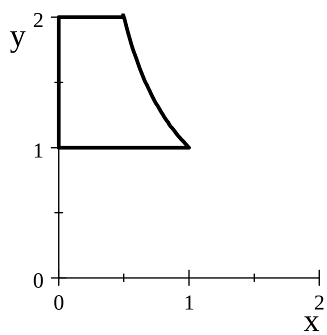
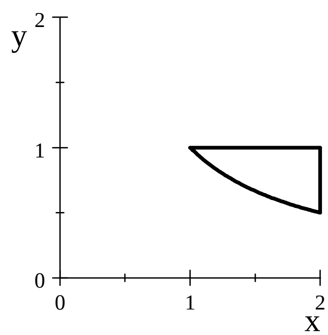
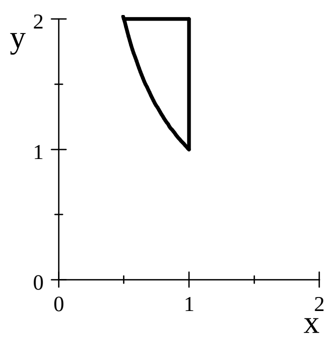


d.


e.


CorrectChoice

5. Find the mass of the quarter circle $x^2 + y^2 \le 9$ for $x \ge 0$ and $y \ge 0$ if the density is $\rho = \sqrt{x^2 + y^2}$.

a. $\frac{9}{4}\pi$

b. $\frac{9}{2}\pi$ Correct Choice

c. $\frac{27}{4}\pi$

d. $\frac{81}{4}\pi$

e. $\frac{243}{2}\pi$

$$M = \iint \rho\, dA = \int_0^{\pi/2}\int_0^3 r\,r\,dr\,d\theta = \frac{\pi}{2}\frac{r^3}{3}\Big|_0^3 = \frac{9}{2}\pi$$

6. Find the center of mass of the quarter circle $x^2 + y^2 \le 9$ for $x \ge 0$ and $y \ge 0$ if the density is $\rho = \sqrt{x^2 + y^2}$.

a. $\left(\frac{4\pi}{9}, \frac{4\pi}{9}\right)$

b. $\left(\frac{2\pi}{9}, \frac{2\pi}{9}\right)$

c. $\left(\frac{\pi}{9}, \frac{\pi}{9}\right)$

d. $\left(\frac{9}{2\pi}, \frac{9}{2\pi}\right)$ Correct Choice

e. $(9,9)$

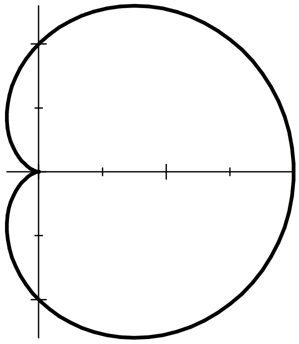
$$M_y = \iint x\rho\, dA = \int_0^{\pi/2}\int_0^3 r\cos\theta\, r\, r\,dr\,d\theta = \frac{r^4}{4}\Big|_0^3\Big[\sin\theta\Big]_0^{\pi/2} = \frac{81}{4} \qquad \bar{x} = \frac{M_y}{M} = \frac{81}{4}\frac{2}{9\pi} = \frac{9}{2\pi}$$

$$M_x = \iint y\rho\, dA = \int_0^{\pi/2}\int_0^3 r\sin\theta\, r\, r\,dr\,d\theta = \frac{r^4}{4}\Big|_0^3\Big[-\cos\theta\Big]_0^{\pi/2} = \frac{81}{4} \qquad \bar{y} = \frac{M_x}{M} = \frac{81}{4}\frac{2}{9\pi} = \frac{9}{2\pi}$$

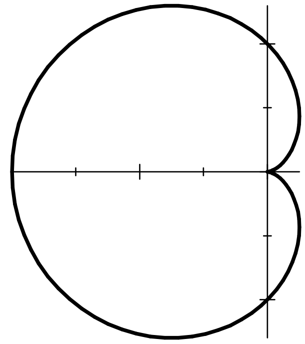
You can skip one of these since $\bar{x} = \bar{y}$ by symmetry.

7. Which of the following is the polar graph of the polar curve $r = 1 - \sin\theta$?

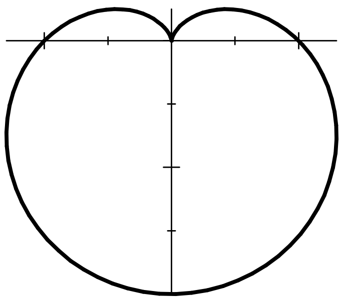
a.



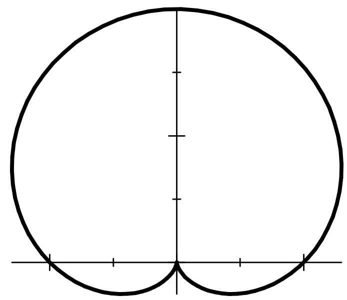
b.



c.

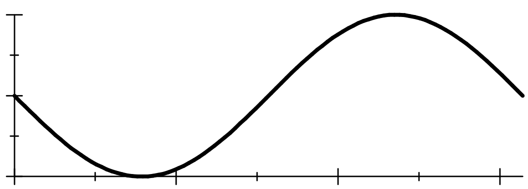


d.



⇑ CorrectChoice ⇑

e.



(e) is the rectangular graph.

(c) is the polar graph.

8. The function $f(x,y) = x\sin 2y - y\cos 2x$ satisfies which differential equation?

a. $\vec{\nabla}\cdot\vec{\nabla}f = -4f$ Correct Choice

b. $\vec{\nabla}\cdot\vec{\nabla}f = -2f$

c. $\vec{\nabla}\cdot\vec{\nabla}f = 0$

d. $\vec{\nabla}\cdot\vec{\nabla}f = 2f$

e. $\vec{\nabla}\cdot\vec{\nabla}f = 4f$

$\vec{\nabla}f = (\sin 2y + 2y\sin 2x, 2x\cos 2y - \cos 2x)$

$\vec{\nabla}\cdot\vec{\nabla}f = 4y\cos 2x - 4x\sin 2y = -4(x\sin 2y - y\cos 2x) = -4f$

9. Find the volume above the cone $z = 2\sqrt{x^2 + y^2}$ below the paraboloid $z = 8 - x^2 - y^2$.

a. $\frac{40}{3}\pi$ Correct Choice

b. 16π

c. $\frac{56}{3}\pi$

d. $\frac{80}{3}\pi$

e. 32π

In cylindrical coordinates, the equations are $z = 2r$ and $z = 8 - r^2$.

They intersect when $2r = 8 - r^2$ or $r^2 + 2r - 8 = 0$ or $(r-2)(r+4) = 0$ or $r = 2$.

$V = \iiint 1\,dV = \int_0^{2\pi}\int_0^2\int_{2r}^{8-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\Big[rz\Big]_{2r}^{8-r^2} dr\,d\theta = \int_0^{2\pi}\int_0^2 r(8 - r^2 - 2r)\,dr\,d\theta$

You could have started from the last integral as polar coordinates.

$V = 2\pi\left[4r^2 - \frac{r^4}{4} - \frac{2r^3}{3}\right]_0^2 = 2\pi\left(16 - 4 - \frac{16}{3}\right) = \frac{40}{3}\pi$

10. Find the average value of the function $f(x,y,z) = z^2$

on the hemisphere $x^2 + y^2 + z^2 \le 4$ for $z \ge 0$. HINT: $f_{\text{ave}} = \frac{1}{V}\iiint f\,dV$

a. 0

b. $\frac{4}{5}$ Correct Choice

c. $\frac{8}{5}$

d. $\frac{\pi}{4}$

e. $\frac{\pi}{2}$

The volume is $V = \frac{1}{2}\cdot\frac{4}{3}\pi 2^3 = \frac{16}{3}\pi$

$\iiint f\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (\rho\cos\varphi)^2\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\left[\frac{-\cos^3\varphi}{3}\right]_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_0^2 = 2\pi\left(\frac{1}{3}\right)\left(\frac{32}{5}\right) = \frac{64}{15}\pi$

$f_{\text{ave}} = \frac{3}{16\pi}\cdot\frac{64\pi}{15} = \frac{4}{5}$

11. Compute $\int \vec{F} \cdot d\vec{s}$ counterclockwise around the circle $x^2 + y^2 = 4$ with $z = 4$ for the vector field $\vec{F} = (-yz, xz, z^2)$.

- **a**. 2π
- **b**. 4π
- **c**. 8π
- **d**. 16π
- **e**. 32π Correct Choice

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4)$ $\quad \vec{v} = (-2\sin\theta, 2\cos\theta, 0)$ $\quad \vec{F}(\vec{r}(\theta)) = (-8\sin\theta, 8\cos\theta, 16)$

$$\int \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (16\sin^2\theta + 16\cos^2\theta)\, d\theta = \int_0^{2\pi} 16\, d\theta = 32\pi$$

12. Compute $\iint \frac{1}{x}\, dS$ on the parametric surface $\vec{R}(u,v) = (u^2 + v^2, u^2 - v^2, 2uv)$ for $1 \le u \le 3$ and $1 \le v \le 4$.

- **a**. $6\sqrt{2}$
- **b**. $12\sqrt{2}$
- **c**. $24\sqrt{2}$ Correct Choice
- **d**. $64\sqrt{2}$
- **e**. $272\sqrt{2}$

$$\begin{array}{r} \\ \vec{e}_u = \\ \vec{e}_v = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (2u & 2u & 2v) \\ (2v & -2v & 2u) \end{array} \right| \qquad \vec{N} = (4u^2 + 4v^2, 4v^2 - 4u^2, -8uv)$$

$$\left|\vec{N}\right| = \sqrt{(4u^2 + 4v^2)^2 + (4v^2 - 4u^2)^2 + (-8uv)^2} = \sqrt{32u^4 + 64u^2v^2 + 32v^4}$$

$$= \sqrt{32(u^4 + 2u^2v^2 + v^4)} = 4\sqrt{2}\,(u^2 + v^2)$$

$$\frac{1}{x} = \frac{1}{u^2 + v^2}$$

$$\iint \frac{1}{x}\, dS = \int_1^4 \int_1^3 \frac{1}{u^2 + v^2} 4\sqrt{2}\,(u^2 + v^2)\, du\, dv = 4\sqrt{2} \int_1^4 \int_1^3 1\, du\, dv = 24\sqrt{2}$$

Work Out: (15 points each. Part credit possible. Show all work.)

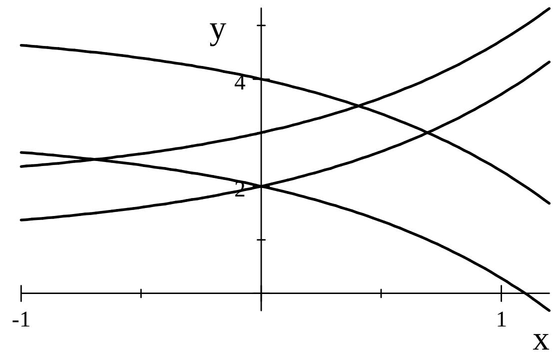
13. A plate has the shape of the region between the curves

$$y = 1 + \frac{1}{2}e^x, \quad y = 2 + \frac{1}{2}e^x,$$
$$y = 3 - \frac{1}{2}e^x, \quad y = 5 - \frac{1}{2}e^x$$

where x and y are measured in centimeters. If the mass density is $\rho = ye^x$ gm/cm^2, find the total mass of the plate.

HINT: Use the curvilinear coordinates

$$u = y - \frac{1}{2}e^x \quad \text{and} \quad v = y + \frac{1}{2}e^x$$



$$u + v = 2y \qquad y = \frac{1}{2}u + \frac{1}{2}v \qquad v - u = e^x \qquad x = \ln(v-u) \qquad \vec{R}(u,v) = \left(\ln(v-u), \frac{1}{2}u + \frac{1}{2}v\right)$$

The boundaries are: $u = 1$, $u = 2$, $v = 3$, $v = 5$

$$\vec{e}_u = \left(\frac{-1}{v-u}, \frac{1}{2}\right) \qquad \vec{e}_v = \left(\frac{1}{v-u}, \frac{1}{2}\right) \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{-1}{v-u} & \frac{1}{2} \\ \frac{1}{v-u} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{-1}{2(v-u)} - \frac{1}{2(v-u)} = \frac{-1}{v-u}$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{-1}{v-u}\right| = \frac{1}{v-u} \quad \text{since } v > u \text{ always.}$$

$$\rho = ye^x = \left(\frac{1}{2}u + \frac{1}{2}v\right)(v-u)$$

The mass is

$$M = \iint \rho\, dA = \iint \rho J\, du\, dv = \int_3^5 \int_1^2 \left(\frac{1}{2}u + \frac{1}{2}v\right)(v-u)\frac{1}{(v-u)}\, du\, dv = \frac{1}{2}\int_3^5 \int_1^2 (u+v)\, du\, dv$$

$$= \frac{1}{2}\int_3^5 \left[\frac{u^2}{2} + vu\right]_{u=1}^2 dv = \frac{1}{2}\int_3^5 \left(\frac{3}{2} + v\right) dv = \frac{1}{2}\left[\frac{3}{2}v + \frac{v^2}{2}\right]_{v=3}^5$$

$$= \frac{1}{2}\left(\frac{15}{2} + \frac{25}{2}\right) - \frac{1}{2}\left(\frac{9}{2} + \frac{9}{2}\right) = \frac{11}{2}$$

14. Find the point in the first octant on the graph of $z = \frac{8}{x^2 y}$ closest to the origin.

Minimize the square of the distance $f = D^2 = x^2 + y^2 + z^2$ subject to the constraint $z = \frac{8}{x^2 y}$.

$$f = x^2 + y^2 + \frac{64}{x^4 y^2}$$

$f_x = 2x - \frac{256}{x^5 y^2} = 0$ $\quad f_y = 2y - \frac{128}{x^4 y^3} = 0$ or $x^6 y^2 = 128$ $\quad x^4 y^4 = 64$

So $y = \frac{\sqrt[4]{64}}{x} = \frac{\sqrt{8}}{x}$ and $x^6 \frac{8}{x^2} = 128$ or $x^4 = 16$

So $x = 2$ $\quad y = \frac{\sqrt{8}}{2} = \sqrt{2}$ $\quad z = \frac{8}{2^2\sqrt{2}} = \sqrt{2}$

15. Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the paraboloid $z = x^2 + y^2$ with $z \le 4$ oriented down and out, for the vector field $\vec{F} = (-yz, xz, z^2)$.

HINT: The paraboloid may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\cos\theta & \sin\theta & 2r) \\ (-r\sin\theta & r\cos\theta & 0) \end{vmatrix} \qquad \vec{N} = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$ This is up and in.

Reverse $\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(-x) - \hat{\jmath}(y) + \hat{k}(z+z) = (-x, -y, 2z)$$

$$\vec{\nabla} \times \vec{F}\left(\vec{R}(r,\theta)\right) = (-r\cos\theta, -r\sin\theta,, 2r^2)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -2r^3\cos^2\theta - 2r^3\sin^2\theta - 2r^3 = -4r^3$$

$$\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 -4r^3\, dr\, d\theta = -2\pi\left[r^4\right]_0^2 = -32\pi$$