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| 1-13 | /65 |
| 14 | /25 |
| 15 | /15 |
| Total | /105 |

MATH 253 Final Exam Spring 2008

Sections 200,501,502 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

**1**. Find the equation of the plane containing the two lines:
$\vec{r}_1(s) = (2+3s, -4-2s, 3-s)$ and $\vec{r}_2(t) = (2-t, -4+2t, 3+2t)$

**a**. $-2x - 5y + 4z = 45$

**b**. $-2x - 5y + 4z = 1$

**c**. $-2x - 5y + 4z = -3$

**d**. $\vec{R}(s,t) = (2+3s-t, -4-2s+2t, 3-s+2t)$ Correct Choice

**e**. $\vec{R}(s,t) = (-2+3s-t, -5-2s+2t, 4-s+2t)$

The tangent vectors to the lines and hence the plane are $\vec{v}_1 = (3,-2,-1)$ and $\vec{v}_2 = (-1,2,2)$.
The point $P = (2,-4,3)$ is on both lines and the plane.
So the parametric equation of the plane is
$\vec{R}(s,t) = P + s\vec{v}_1 + t\vec{v}_2 = (2+3s-t, -4-2s+2t, 3-s+2t)$
(The normal equation is $-2x - 5y + 4z = 28$.)

**2**. Find the equation of the plane tangent to the graph of the function
$f(x,y) = x^2 + xy + y^2$ at the point $(2,3)$. Then the $z$-intercept is

**a**. $-38$

**b**. $-19$ Correct Choice

**c**. $0$

**d**. $19$

**e**. $38$

$f = x^2 + xy + y^2$ $f(2,3) = 19$ $z = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3)$

$f_x = 2x + y$ $f_x(2,3) = 7$ $= 19 + 7(x-2) + 8(y-3)$

$f_y = x + 2y$ $f_y(2,3) = 8$ $= 7x + 8y - 19$

**3.** Find the arc length of the curve $\vec{r}(t) = (\ln t, 2t, t^2)$ between $(0,2,1)$ and $(1,2e,e^2)$.

Hint: Look for a perfect square.

- **a.** $e^2$ Correct Choice
- **b.** $1 + e^2$
- **c.** $e^2 - 1$
- **d.** $2 + e^2$
- **e.** $e^2 - 2$

$$\vec{v} = \left(\frac{1}{t}, 2, 2t\right) \qquad |\vec{v}| = \sqrt{\frac{1}{t^2} + 4 + 4t^2} = \frac{1}{t} + 2t$$

$\vec{r}(t) = (0,2,1)$ at $t = 1$ $\qquad$ $\vec{r}(t) = (1,2e,e^2)$ at $t = e$

$$L = \int_1^e |\vec{v}|\,dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt = \left[\ln t + t^2\right]_1^e = (\ln e + e^2) - (\ln 1 + 1) = e^2$$

**4.** Find the unit binormal $\hat{B}$ of the curve $\vec{r}(t) = (\ln t, 2t, t^2)$ at $t = 1$.

Hint: Plug $t = 1$ into $\vec{v}$ and $\vec{a}$.

- **a.** $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$
- **b.** $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$
- **c.** $\left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$
- **d.** $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$
- **e.** $\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ Correct Choice

$\vec{v} = \left(\frac{1}{t}, 2, 2t\right) \qquad \vec{a} = \left(-\frac{1}{t^2}, 0, 2\right)$ $\qquad$ At $t = 1$: $\qquad \vec{v} = (1,2,2) \qquad \vec{a} = (-1,0,2) \qquad \vec{v} \times \vec{a} = (4,-4,2)$

$$|\vec{v} \times \vec{a}| = \sqrt{16 + 16 + 4} = 6 \qquad \hat{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{1}{6}(4,-4,2) = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

**5.** The volume of a square pyramid is $V = \frac{1}{3}s^2h$.
If the side of the base $s$ is currently $3$ cm and increasing at $2$ cm/sec
while the height $h$ is currently $4$ cm and decreasing at $1$ cm/sec,
is the volume increasing or decreasing and at what rate?

**a.** increasing at $19$ cm$^3/\text{sec}$

**b.** increasing at $13$ cm$^3/\text{sec}$ Correct Choice

**c.** neither increasing nor decreasing

**d.** decreasing at $13$ cm$^3/\text{sec}$

**e.** decreasing at $19$ cm$^3/\text{sec}$

$$\frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2}{3}sh\frac{ds}{dt} + \frac{1}{3}s^2\frac{dh}{dt} = \frac{2}{3}(3)(4)(2) + \frac{1}{3}(3)^2(-1) = 13$$

This is positive and so increasing.

**6.** Which of the following is a local minimum of $f(x,y) = \sin(x)\cos(y)$?

**a.** $(0,0)$

**b.** $\left(\frac{\pi}{2},0\right)$

**c.** $(\pi,\pi)$

**d.** $\left(0,\frac{\pi}{2}\right)$

**e.** None of the above Correct Choice

$f_x(x,y) = \cos(x)\cos(y) \qquad f_y(x,y) = -\sin(x)\sin(y)$

Since $f_x(0,0) = f_x(\pi,\pi) = 1$, the points $(0,0)$ and $(\pi,\pi)$ are not even critical points.

$f_{xx}(x,y) = -\sin(x)\cos(y) \qquad f_{yy}(x,y) = -\sin(x)\cos(y) \qquad f_{xy}(x,y) = -\cos(x)\sin(y)$

Since $f_{xx}\left(\frac{\pi}{2},0\right) = -1 < 0 \qquad f_{yy}\left(\frac{\pi}{2},0\right) = -1 < 0 \qquad f_{xy}\left(\frac{\pi}{2},0\right) = 0$

we have $D\left(\frac{\pi}{2},0\right) = f_{xx}f_{yy} - f_{xy}^2 = 1$ and $\left(\frac{\pi}{2},0\right)$ is a local maximum.

Since $f_{xx}\left(0,\frac{\pi}{2}\right) = 0 \qquad f_{yy}\left(0,\frac{\pi}{2}\right) = 0 \qquad f_{xy}\left(0,\frac{\pi}{2}\right) = -1$

we have $D\left(0,\frac{\pi}{2}\right) = f_{xx}f_{yy} - f_{xy}^2 = -1$ and $\left(0,\frac{\pi}{2}\right)$ is a saddle.

**7**. Find the equation of the plane tangent to the surface $x^2z^2 + yz^3 = 11$ at the point $(2,3,1)$.
Then the intersection with the $x$-axis is at

- **a**. $(28,0,0)$
- **b**. $(16,0,0)$
- **c**. $(14,0,0)$
- **d**. $(7,0,0)$ Correct Choice
- **e**. $(4,0,0)$

Let $f = x^2z^2 + yz^3$ and $P = (2,3,1)$. Then $\vec{\nabla}f = (2xz^2, z^3, 2x^2z + 3yz^2)$ and $\vec{N} = \vec{\nabla}f\Big|_P = (4,1,17)$

$\vec{N} \cdot X = \vec{N} \cdot P$ $4x + y + 17z = 4 \cdot 2 + 3 + 17 \cdot 1 = 28$ If $y = z = 0$, then $x = 7$.

**8**. Compute $\int \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = (y,x)$

along the curve $\vec{r}(t) = (t + \sin t, t + \cos t)$ from $\vec{r}(\pi)$ to $\vec{r}(2\pi)$.

Hint: Find a scalar potential.

- **a**. $3\pi^2 + 3\pi$ Correct Choice
- **b**. $3\pi^2 - 3\pi$
- **c**. $3\pi^2 + \pi$
- **d**. $3\pi^2 - \pi$
- **e**. $3\pi - 3\pi^2$

$\vec{F} = (y,x) = \vec{\nabla}f$ where $f = xy$. $\vec{r}(\pi) = (\pi, \pi - 1)$ $\vec{r}(2\pi) = (2\pi, 2\pi + 1)$

$\int \vec{F} \cdot d\vec{s} = \int \vec{\nabla}f \cdot d\vec{s} = f(2\pi, 2\pi + 1) - f(\pi, \pi - 1) = 2\pi(2\pi + 1) - \pi(\pi - 1) = 3\pi^2 + 3\pi$

**9**. Find the mass of the solid hemisphere $x^2 + y^2 + z^2 \leq 4$ for $y \geq 0$
if the density is $\delta = z^2$.

**a**. $\frac{4}{3}\pi^2$

**b**. $\frac{8}{3}\pi^2$

**c**. $\frac{32\pi}{15}$

**d**. $\frac{64\pi}{15}$ Correct Choice

**e**. $\frac{128\pi}{15}$

$$M = \iiint \delta\, dV = \int_0^{\pi}\int_0^{\pi}\int_0^{2} \rho^2\cos^2\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = (\pi)\left[\frac{-\cos^3\varphi}{3}\right]_0^{\pi}\left[\frac{r^5}{5}\right]_0^{2} = \pi \cdot \frac{2}{3} \cdot \frac{32}{5} = \frac{64\pi}{15}$$

**10**. Find the center of mass of the solid hemisphere $x^2 + y^2 + z^2 \leq 4$ for $y \geq 0$
if the density is $\delta = z^2$.

**a**. $\left(0, \frac{5}{8}, 0\right)$ Correct Choice

**b**. $\left(0, \frac{8}{5}, 0\right)$

**c**. $\left(0, \frac{8\pi}{3}, 0\right)$

**d**. $\left(0, \frac{3}{8\pi}, 0\right)$

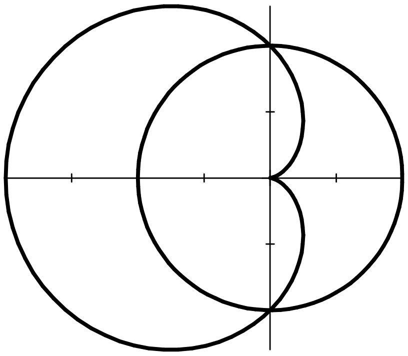
**e**. $\left(0, \frac{3}{4\pi}, 0\right)$

$$M_{xz} = \iiint y\delta\, dV = \int_0^{\pi}\int_0^{\pi}\int_0^{2} \rho\sin\varphi\sin\theta\rho^2\cos^2\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{\pi}\sin\theta\, d\theta\int_0^{\pi}\cos^2\varphi\sin^2\varphi\, d\varphi\int_0^{2}\rho^5\, d\rho$$

$$= \left[-\cos\theta\right]_0^{\pi}\int_0^{\pi}\frac{\sin^2(2\varphi)}{4}\, d\varphi\left[\frac{r^6}{6}\right]_0^{2} = \frac{2^6}{3}\int_0^{\pi}\frac{1-\cos(4\varphi)}{8}\, d\varphi = \frac{2^3}{3}\left[\varphi - \frac{\sin(4\varphi)}{4}\right]_0^{\pi} = \frac{8\pi}{3}$$

$\bar{y} = \frac{M_{xz}}{M} = \frac{8\pi}{3}\,\frac{15}{64\pi} = \frac{5}{8}$ By symmetry, $\bar{x} = 0$ $\bar{z} = 0$

**11**. Find the area inside the circle $r = 1$ but outside the cardioid $r = 1 - \cos\theta$.



**a**. $\frac{\pi}{4}$

**b**. $\frac{\pi}{2}$

**c**. $2 - \frac{\pi}{4}$ Correct Choice

**d**. $2 + \frac{\pi}{4}$

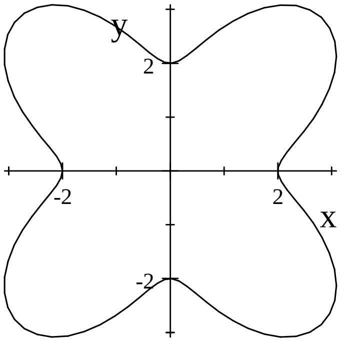
**e**. $2 - \frac{\pi}{2}$

$$A = \iint 1\,dA = \int_{-\pi/2}^{\pi/2}\int_{1-\cos\theta}^{1} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=1-\cos\theta}^{1} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(1-[1-\cos\theta]^2\right)d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}(2\cos\theta - \cos^2\theta)\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(2\cos\theta - \frac{1+\cos(2\theta)}{2}\right)d\theta$$

$$= \frac{1}{2}\left[2\sin\theta - \frac{1}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\right]_{-\pi/2}^{\pi/2} = \frac{1}{2}\left[2(1) - \frac{1}{2}\left(\frac{\pi}{2}\right)\right] - \frac{1}{2}\left[2(-1) - \frac{1}{2}\left(\frac{-\pi}{2}\right)\right] = 2 - \frac{\pi}{4}$$

**12**. Compute $\oint \vec{\nabla} f \cdot \vec{ds}$ counterclockwise once around the polar curve $r = 3 - \cos(4\theta)$ for the function $f(x,y) = x^2 y$.



**a**. $2\pi$

**b**. $4\pi$

**c**. $6\pi$

**d**. $8\pi$

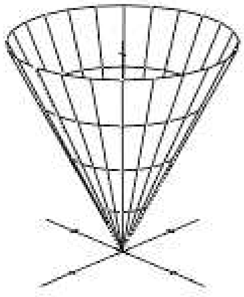
**e**. $0$ Correct Choice

By the FTCC, $\int_A^B \vec{\nabla} f \cdot \vec{ds} = f(B) - f(A)$.

However, since it is a closed curve, $B = A$, and $\int_A^B \vec{\nabla} f \cdot \vec{ds} = 0$.

OR by Green's Theorem, $\oint \vec{\nabla} f \cdot \vec{ds} = \iint \vec{\nabla} \times \vec{\nabla} f \cdot \hat{k}\,dA = 0$ because $\vec{\nabla} \times \vec{\nabla} f = 0$ for any $f$.

**13.** Stokes' Theorem states $\iint\limits_{C} \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint\limits_{\partial C} \vec{F} \cdot d\vec{s}$



Compute either integral for the cone $C$ given by

$z = 2\sqrt{x^2 + y^2}$ for $z \le 8$ oriented up and in,

and the vector field $\vec{F} = (yz, -xz, z)$.

Note: The cone may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2r)$

The boundary of the cone is the circle $x^2 + y^2 = 16$ with $z = 8$.

**a.** $-768\pi$
**b.** $-256\pi$ Correct Choice

**c.** $64\pi$

**d.** $256\pi$

**e.** $768\pi$

The surface integral:

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta & \sin\theta, & 2) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(-2r\cos\theta) - \hat{\jmath}(2r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$
$$= (-2r\cos\theta, -2r\sin\theta, r) \quad \text{oriented correctly}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz, & -xz, & z \end{vmatrix} = \hat{\imath}(0 - -x) - \hat{\jmath}(0 - y) + \hat{k}(-z - z) = (x, y, -2z) = (r\cos\theta, r\sin\theta, -4r)$$

$$\iint\limits_{C} \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint\limits_{C} \vec{\nabla} \times \vec{F} \cdot \vec{N}\, du\, dv = \int_0^{2\pi} \int_0^4 (-2r^2\cos^2\theta - 2r^2\sin^2\theta - 4r^2)\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^4 (-6r^2)\, dr\, d\theta = 2\pi[-2r^3]_0^4 = -256\pi$$

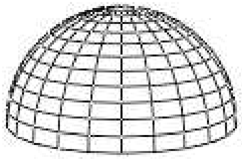
The line integral: $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 8)$ $\vec{v} = (-4\sin\theta, 4\cos\theta, 0)$

$\vec{F}(\vec{r}(\theta)) = (32\sin\theta, -32\cos\theta, 8)$

$$\oint\limits_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (-128\sin^2\theta - 128\cos^2\theta)\, d\theta = \int_0^{2\pi} (-128)\, d\theta = -256\pi$$

Work Out: (Part credit possible. Show all work.)

**14.** (25 points) Verify Gauss' Theorem $\iiint\limits_H \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial H} \vec{F} \cdot d\vec{S}$



for the solid hemisphere $x^2 + y^2 + z^2 \le 4$ with $z \ge 0$

and the vector field $\vec{F} = (xz^2, yz^2, x^2 + y^2)$.

Notice that the boundary of the solid hemisphere $\partial H$ consists of the hemisphere surface $S$ given by $x^2 + y^2 + z^2 = 4$ with $z \ge 0$ and the disk $D$ given by $x^2 + y^2 \le 4$ with $z = 0$.

Be sure to check and explain the orientations. Use the following steps:

**a.** Compute the volume integral by successively finding:

$$\vec{\nabla} \cdot \vec{F}(x,y,z), \quad \vec{\nabla} \cdot \vec{F}(\rho,\theta,\varphi), \quad dV, \quad \iiint\limits_H \vec{\nabla} \cdot \vec{F}\, dV$$

$$\vec{\nabla} \cdot \vec{F} = z^2 + z^2 + 0 = 2z^2 = 2\rho^2 \cos^2\varphi \qquad dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$\iiint\limits_H \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^2 2\rho^2 \cos^2\varphi\, \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi = 2\pi \left[ \frac{2\rho^5}{5} \right]_0^2 \left[ \frac{-\cos^3\varphi}{3} \right]_0^{\pi/2}$$

$$= \frac{128\pi}{5} \left( \frac{-0}{3} - \frac{-1}{3} \right) = \frac{128\pi}{15}$$

**b.** Compute the surface integral over the disk by parametrizing the disk and successively finding:

$$\vec{R}(r,\theta), \quad \vec{e}_r, \quad \vec{e}_\theta, \quad \vec{N}, \quad \vec{F}\big(\vec{R}(r,\theta)\big), \quad \iint\limits_D \vec{F} \cdot d\vec{S}$$

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 0) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (0,0,r)$$

We need $\vec{N}$ to point down (out of the volume). Reverse $\vec{N} = (0,0,-r)$

$$\vec{F}\big(\vec{R}(r,\theta)\big) = (0,0,r^2)$$

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 -r^3\, dr\, d\theta = -2\pi \left[ \frac{r^4}{4} \right]_0^2 = -8\pi$$

Recall: $\vec{F} = (xz^2, yz^2, x^2 + y^2)$

**c.** Compute the surface integral over the hemisphere by parametrizing the surface and successively finding:

$$\vec{R}(\theta,\varphi), \quad \vec{e}_\theta, \quad \vec{e}_\varphi, \quad \vec{N}, \quad \vec{F}\left(\vec{R}(\theta,\varphi)\right), \quad \iint_S \vec{F} \cdot d\vec{S}$$

$$\vec{R}(\theta,\varphi) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta = & (-2\sin\varphi\sin\theta, & 2\sin\varphi\cos\theta, & 0 \quad ) \\ \vec{e}_\varphi = & (2\cos\varphi\cos\theta, & 2\cos\varphi\sin\theta, & -2\sin\varphi) \end{array}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_\varphi = \hat{\imath}(-4\sin^2\varphi\cos\theta) - \hat{\jmath}(4\sin^2\varphi\sin\theta) + \hat{k}(-4\sin\varphi\cos\varphi\sin^2\theta - 4\sin\varphi\cos\varphi\cos^2\theta)$$
$$= (-4\sin^2\varphi\cos\theta, -4\sin^2\varphi\sin\theta, -4\sin\varphi\cos\varphi)$$

We need $\vec{N}$ to point up (out of the volume). Reverse $\vec{N} = (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$

$$\vec{F}\left(\vec{R}(r,\theta)\right) = (8\sin\varphi\cos^2\varphi\cos\theta, 8\sin\varphi\cos^2\varphi\sin\theta, 4\sin^2\varphi)$$

$$\vec{F} \cdot \vec{N} = 32\sin^3\varphi\cos^2\varphi\cos^2\theta + 32\sin^3\varphi\cos^2\varphi\sin^2\theta + 16\sin^3\varphi\cos\varphi = 32\sin^3\varphi\cos^2\varphi + 16\sin^3\varphi\cos\varphi$$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 32\sin^3\varphi\cos^2\varphi + 16\sin^3\varphi\cos\varphi\, d\varphi\, d\theta$$
$$= 2\pi\int_0^{\pi/2} 32\sin\varphi(1-\cos^2\varphi)\cos^2\varphi\, d\varphi + 2\pi\int_0^{\pi/2} 16\sin^3\varphi\cos\varphi\, d\varphi$$

First integral: $u = \cos\varphi \qquad du = -\sin\varphi\, d\varphi$

Second integral: $v = \sin\varphi \qquad dv = \cos\varphi\, d\varphi$

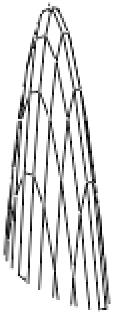
$$\iint_S \vec{F} \cdot d\vec{S} = -2\pi\int_1^0 32(1-u^2)u^2\, du + 2\pi\int_0^1 16v^3\, dv = -64\pi\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_1^0 + 32\pi\left[\frac{v^4}{4}\right]_0^1$$
$$= 64\pi\left[\frac{1}{3} - \frac{1}{5}\right] + 32\pi\left[\frac{1}{4}\right] = \frac{128\pi}{15} + 8\pi = \frac{248\pi}{15}$$

**d.** Combine $\iint_D \vec{F} \cdot d\vec{S}$ and $\iint_S \vec{F} \cdot d\vec{S}$ to get $\iint_{\partial H} \vec{F} \cdot d\vec{S}$

$$\iint_{\partial H} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} + \iint_S \vec{F} \cdot d\vec{S} = \frac{128\pi}{15} + 8\pi - 8\pi = \frac{128\pi}{15}$$

which agrees with part (a).

**15**. (15 points) A rectangular solid sits on the $xy$-plane with its top four vertices on the paraboloid $z = 9 - 9x^2 - y^2$. Find the dimensions and volume of the largest such box.



The dimensions are $2x$, $2y$ and $z$.

$V = (2x)(2y)z = 4xyz = 4xy(9 - 9x^2 - y^2) = 36xy - 36x^3y - 4xy^3$

$V_x = 36y - 108x^2y - 4y^3 = 0$ $\quad V_y = 36x - 36x^3 - 12xy^2 = 0$

$x > 0$, $y > 0$ and $z > 0$ to give positive, non-zero dimensions.

(1) $9 - 27x^2 - y^2 = 0$ $\quad$ (2) $3 - 3x^2 - y^2 = 0$

(1) - (2): $6 - 24x^2 = 0$ $\quad x^2 = \frac{1}{4}$ $\quad x = \frac{1}{2}$

$y^2 = 3 - 3x^2 = 3 - \frac{3}{4} = \frac{9}{4}$ $\quad y = \frac{3}{2}$

$z = 9 - 9x^2 - y^2 = 9 - \frac{9}{4} - \frac{9}{4} = \frac{9}{2}$

The dimensions are $2x = 1$, $2y = 3$ and $z = \frac{9}{2}$. $\quad V = (2x)(2y)z = \frac{27}{2}$

**16**. (5 points) (Honors only. Replaces #2.) Find the plane tangent to the parametric surface $\vec{R}(u,v) = (u+v, u-v, uv)$ at the point $\vec{R}(1,1) = (2,0,1)$.
Give both the parametric equation and the normal equation of the tangent plane.

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (1 & 1, & v) \\ \vec{e}_v = & (1, & -1, & u) \end{array}$$

$\vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(u+v) - \hat{\jmath}(u-v) + \hat{k}(-1-1) = (u+v, v-u, -2)$

At $P = \vec{R}(1,1) = (2,0,1)$, $\vec{e}_u = (1,1,1)$, $\vec{e}_v = (1,-1,1)$, $\vec{N} = (2,0,-2)$

So the parametric equation is

$X = P + s\vec{e}_u + t\vec{e}_v \qquad (x,y,z) = (2,0,1) + s(1,1,1) + t(1,-1,1) = (2+s+t, s-t, 1+s+t)$

and the normal equation is

$\vec{N} \bullet X = \vec{N} \bullet P \qquad 2x - 2z = 2 \bullet 2 - 2 \bullet 1 = 2$ or $x - z = 1$