Name______________________________ Sec__________

1-13	/52	15	/15
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Total			/107

MATH 253 Exam 2 Fall 2008

Sections 501-503,200 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. The point $(2,3)$ is a critical point of the function $f = xy + \frac{12}{x} + \frac{18}{y}$.

Apply the Second Derivative Test to classify the point $(2,3)$.

a. local minimum correct choice

b. local maximum

c. inflection point

d. saddle point

e. Test Fails

$f_x = y - \frac{12}{x^2}$ $\quad f_x(2,3) = 0$ $\qquad f_y = x - \frac{18}{y^2}$ $\quad f_y(2,3) = 0$

$f_{xx} = \frac{24}{x^3}$ $\quad f_{yy} = \frac{36}{y^3}$ $\quad f_{xy} = 1$ $\qquad f_{xx}(2,3) = 3 > 0$ $\quad f_{yy}(2,3) = \frac{36}{27} = \frac{4}{3}$ $\quad f_{xy}(2,3) = 1$

$D(2,-3) = f_{xx}f - f_{xy}^2 = (3)\left(\frac{4}{3}\right) - (1)^2 = 3 > 0$ local minimum

2. Find the volume below the function $z = xy$ above the region in the xy-plane bounded by $x = 4y$ and $x = y^2$.

a. $\frac{80}{3}$

b. $\frac{160}{3}$

c. $\frac{512}{3}$ correct choice

d. $\frac{640}{3}$

e. $\frac{1024}{3}$

$$V = \int_0^4 \int_{y^2}^{4y} xy\,dx\,dy = \int_0^4 \left[\frac{x^2 y}{2}\right]_{x=y^2}^{4y} dy = \int_0^4 (8y^3) - \left(\frac{y^5}{2}\right) dy = \left[2y^4 - \frac{y^6}{12}\right]_0^4 = 2(4)^4 - \frac{(4)^6}{12} = \frac{512}{3}$$

3. Find the mass of the solid below the surface $z = x^2$ above the triangle with vertices $(0,0)$, $(1,0)$ and $(1,2)$, if the density is $\rho = y$.

a. $\frac{1}{5}$

b. $\frac{2}{5}$ correct choice

c. $\frac{4}{5}$

d. $\frac{6}{5}$

e. $\frac{8}{5}$

$$M = \iiint \rho\, dV = \int_0^1 \int_0^{2x} \int_0^{x^2} y\, dz\, dy\, dx = \int_0^1 \int_0^{2x} [yz]_{z=0}^{x^2}\, dy\, dx = \int_0^1 \int_0^{2x} yx^2\, dy\, dx = \int_0^1 \left[x^2 \frac{y^2}{2} \right]_0^{2x} dx$$

$$= \int_0^1 2x^4\, dx = \left[\frac{2x^5}{5} \right]_0^1 = \frac{2}{5}$$

4. Find the area of one leaf of the rose $r = \sin(4\theta)$.

a. π

b. $\frac{\pi}{2}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{8}$

e. $\frac{\pi}{16}$ correct choice

$r = 0$ when $\sin(4\theta) = 0$ or $4\theta = 0, \pi$ or $\theta = 0, \frac{\pi}{4}$

$$A = \iint 1\, dA = \int_0^{\pi/4} \int_0^{\sin(4\theta)} r\, dr\, d\theta = \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^{\sin(4\theta)} d\theta = \frac{1}{2} \int_0^{\pi/4} \sin^2(4\theta)\, d\theta = \frac{1}{2} \int_0^{\pi/4} \frac{1 - \cos(8\theta)}{2}\, d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{\sin(8\theta)}{8} \right]_0^{\pi/4} = \frac{1}{4} \left(\frac{\pi}{4} - \frac{\sin(2\pi)}{8} \right) - \frac{1}{4} \left(0 - \frac{\sin(0)}{8} \right) = \frac{\pi}{16}$$

5. Find the center of mass of the quarter circle $x^2+y^2 \le 9$ in the first quadrant, if the density is $\rho = \sqrt{x^2+y^2}$.

a. $(\bar{x},\bar{y}) = \left(\frac{9}{4}, \frac{9}{4}\right)$

b. $(\bar{x},\bar{y}) = \left(\frac{9}{2}, \frac{9}{2}\right)$

c. $(\bar{x},\bar{y}) = \left(\frac{2}{9}, \frac{2}{9}\right)$

d. $(\bar{x},\bar{y}) = \left(\frac{9}{2\pi}, \frac{9}{2\pi}\right)$ correct choice

e. $(\bar{x},\bar{y}) = \left(\frac{2\pi}{9}, \frac{2\pi}{9}\right)$

$$M = \iint \rho\, dA = \int_0^{\pi/2}\int_0^3 r\, r\, dr\, d\theta = \frac{\pi}{2}\left[\frac{r^3}{3}\right]_0^3 = \frac{9\pi}{2} \qquad \bar{x} = \bar{y} \text{ by symmetry}$$

$$M_y = \iint x\rho\, dA = \int_0^{\pi/2}\int_0^3 r\cos(\theta) r\, r\, dr\, d\theta = [\sin(\theta)]_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^3 = \frac{81}{4} \qquad \bar{x} = \frac{M_y}{M} = \frac{81}{4}\frac{2}{9\pi} = \frac{9}{2\pi}$$

6. Compute $\int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^{x^2+y^2} e^z\, dz\, dy\, dx$. HINT: Convert to cylindrical coordinates.

a. πe^4

b. $\pi(e^4-1)$

c. $\frac{\pi}{2}(e^4-3)$

d. $\pi(e^4-4)$

e. $\frac{\pi}{2}(e^4-5)$ correct choice

$$I = \int_0^{\pi}\int_0^2\int_0^{r^2} e^z\, r\, dz\, dr\, d\theta = \int_0^{\pi}\int_0^2 [e^z]_0^{r^2}\, r\, dr\, d\theta = \int_0^{\pi}\int_0^2 \left(e^{r^2}-1\right) r\, dr\, d\theta = \pi\int_0^2 \left(e^{r^2}r - r\right) dr$$

$$= \pi\left[\frac{1}{2}e^{r^2} - \frac{r^2}{2}\right]_0^2 = \frac{\pi}{2}(e^4-4-1) = \frac{\pi}{2}(e^4-5)$$

7. Find the average value of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ within the sphere $x^2+y^2+z^2 \le 4$.

a. 3

b. 2

c. $\frac{3}{2}$ correct choice

d. $\frac{3}{4}$

e. $\frac{3}{8}$

$$V = \iiint 1\,dV = \int_0^\pi \int_0^{2\pi} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi = \frac{4}{3}\pi r^3 = \frac{32\pi}{3}$$

$$\iiint f dV = \int_0^\pi \int_0^{2\pi} \int_0^2 \rho \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi = (2\pi)[-\cos\varphi]_0^\pi \left[\frac{\rho^4}{4}\right]_0^2 = 16\pi$$

$$f_{\text{ave}} = \frac{\iiint f dV}{V} = 16\pi \frac{3}{32\pi} = \frac{3}{2}$$

8. Find the mass of a wire in the shape of the curve $\vec{r}(t) = \left(e^t, \sqrt{2}\,t, e^{-t}\right)$ for $0 \le t \le 1$ if the linear density is $\rho = x^2$.

a. $\frac{e^3}{3} + e$

b. $\frac{e^3}{3} + e - \frac{4}{3}$ correct choice

c. $\frac{1}{2}e^2$

d. $\frac{1}{2}e^2 + \frac{1}{2}$

e. $\frac{1}{2}e^2 - \frac{1}{2}$

$$\vec{v} = \left(e^t, \sqrt{2}, -e^{-t}\right) \qquad |\vec{v}| = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t} \qquad \rho = e^{2t}$$

$$M = \int \rho \, ds = \int_0^1 e^{2t}(e^t + e^{-t})\,dt = \int_0^1 (e^{3t} + e^t)\,dt = \left[\frac{e^{3t}}{3} + e^t\right]_0^1 = \frac{e^3}{3} + e - \frac{4}{3}$$

9. Compute the line integral $\int_P \vec{\nabla} f \cdot d\vec{s}$

10. for the function $f = xy$ along the parabola $y = x^2$ from the point $(-1,1)$ to the point $(1,1)$.
NOTE: The parabola may be parametrized as $\vec{r}(t) = (t, t^2)$.

a. 0

b. 1

c. 2 correct choice

d. 3

e. 4

$$\vec{v} = (1, 2t) \qquad \vec{\nabla} f = (y, x) = (t^2, t) \qquad \int_P \vec{\nabla} f \cdot ds = \int_{-1}^1 \vec{\nabla} f \cdot \vec{v}\,dt = \int_{-1}^1 (t^2 + 2t^2)\,dt = \left[3\frac{t^3}{3}\right]_{-1}^1 = 2$$

11. Compute the line integral $\oint_C \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = (-x^2 y, xy^2)$ counterclockwise around the circle $x^2 + y^2 = 9$.

a. 324π

b. 162π

c. 81π

d. $\dfrac{81\pi}{2}$ correct choice

e. $\dfrac{81\pi}{4}$

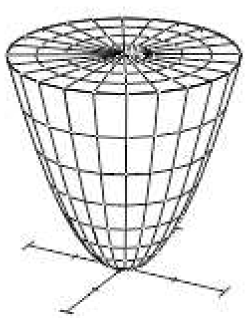
The circle may be parametrized as $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta)$.

$\vec{v} = (-3\sin\theta, 3\cos\theta)$ $\quad \vec{F} = (-27\cos^2\theta\sin\theta, 27\cos\theta\sin^2\theta)$

$$\oint_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (81\cos^2\theta\sin^2\theta + 81\cos^2\theta\sin^2\theta)\, d\theta = 162\int_0^{2\pi} \cos^2\theta\sin^2\theta\, d\theta$$

$$= 162\int_0^{2\pi} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta = \frac{81}{2}\int_0^{2\pi} \sin^2 2\theta\, d\theta = \frac{81}{2}\int_0^{2\pi} \frac{1-\cos 4\theta}{2}\, d\theta = \frac{81}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi} = \frac{81\pi}{2}$$

12. Compute $\iiint_R \vec{\nabla} \cdot \vec{F}\, dV$ for the vector field $\vec{F} = (xz, yz, z^2)$ over the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 4$.



a. $\dfrac{256\pi}{3}$ correct choice

b. $\dfrac{512\pi}{3}$

c. $\dfrac{256\pi}{5}$

d. $\dfrac{512\pi}{5}$

e. $\dfrac{1024\pi}{5}$

In cylindrical coordinates, $r^2 \le z = 4$, $dV = r\,dr\,d\theta\,dz$ and $\vec{\nabla} \cdot \vec{F} = z + z + 2z = 4z$

$$\iiint_R \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 4z\, r\, dz\, dr\, d\theta = 2\pi\int_0^2 [2z^2]_{r^2}^4\, r\, dr = 2\pi\int_0^2 (32 - 2r^4)\, r\, dr$$

$$= 2\pi\left[16r^2 - 2\frac{r^6}{6}\right]_0^2 = 2\pi\left(64 - \frac{64}{3}\right) = \frac{256\pi}{3}$$

13. Find the equation of the plane tangent to the surface $\vec{R}(s,t) = \left(st, \frac{1}{2}s^2 + \frac{1}{2}t^2, \frac{1}{2}s^2 - \frac{1}{2}t^2\right)$ at the point where $(s,t) = (3,1)$.

a. $3x - 5y + 4z = 0$ correct choice

b. $3x + 5y + 4z = 0$

c. $3x - 5y + 4z = 50$

d. $3x + 5y + 4z = 50$

e. $-6x - 10y - 8z = -50$

$$P = \vec{R}(3,1) = \left(3, \frac{1}{2}9 + \frac{1}{2}, \frac{1}{2}9 - \frac{1}{2}\right) = (3,5,4)$$

$$\vec{e}_s = (t,s,s) \qquad \vec{u} = \vec{e}_s(3,1) = (1,3,3) \qquad \vec{e}_t = (s,t,-t) \qquad \vec{v} = \vec{e}_t(3,1) = (3,1,-1)$$

$$\vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 3 \\ 3 & 1 & -1 \end{vmatrix} = \hat{\imath}(-3-3) - \hat{\jmath}(-1-9) + \hat{k}(1-9) = (-6,10,-8)$$

$$\vec{N} \cdot X = \vec{N} \cdot P \qquad -6x + 10y - 8z = -6 \cdot 3 + 10 \cdot 5 - 8 \cdot 4 = 0 \qquad 3x - 5y + 4z = 0$$

14. Find the area of the surface $\vec{R}(s,t) = \left(st, \frac{1}{2}s^2 + \frac{1}{2}t^2, \frac{1}{2}s^2 - \frac{1}{2}t^2\right)$ for $0 \le s \le 1$ and $0 \le t \le 1$.
HINT: Look for a perfect square.

a. $\dfrac{\sqrt{2}}{3}$

b. $\dfrac{1}{3}$

c. $\dfrac{28}{45}\sqrt{2}$

d. $\dfrac{28}{45}$

e. $\dfrac{2\sqrt{2}}{3}$ correct choice

$$\vec{N} = \vec{e}_s \times \vec{e}_t = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t & s & s \\ s & t & -t \end{vmatrix} = \hat{\imath}(-st - st) - \hat{\jmath}(-t^2 - s^2) + \hat{k}(t^2 - s^2) = (-2st, s^2 + t^2, t^2 - s^2)$$

$$\left|\vec{N}\right| = \sqrt{(2st)^2 + (s^2+t^2)^2 + (t^2-s^2)^2} = \sqrt{4s^2t^2 + s^4 + 2s^2t^2 + t^4 + t^4 - 2s^2t^2 + s^4}$$

$$= \sqrt{2s^4 + 4s^2t^2 + 2t^4} = \sqrt{2}\,(s^2 + t^2)$$

$$A = \iint dS = \iint \left|\vec{N}\right| ds\,dt = \int_0^1 \int_0^1 \sqrt{2}\,(s^2+t^2)\,ds\,dt = \sqrt{2}\int_0^1 \left[\frac{s^3}{3} + t^2 s\right]_{s=0}^1 dt$$

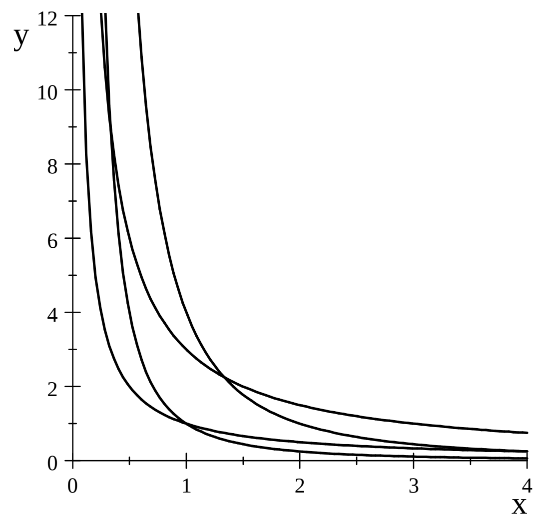
$$= \sqrt{2}\int_0^1 \left(\frac{1}{3} + t^2\right) dt = \sqrt{2}\left[\frac{1}{3}t + \frac{t^3}{3}\right]_{t=0}^1 = \sqrt{2}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2\sqrt{2}}{3}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

15. (15 points) Compute $\iint x^2 y\,dA$ over the "diamond" shaped region in the first quadrant bounded by the curves

$$y = \frac{1}{x} \qquad y = \frac{3}{x} \qquad y = \frac{1}{x^2} \qquad y = \frac{4}{x^2}$$

HINT: Let $u = xy \qquad v = x^2 y$



Solve for x and y: Eliminate $y = \frac{u}{x}$. So $v = x^2 \frac{u}{x} = xu$.

So $x = \frac{v}{u}$. Then $y = \frac{u}{x} = u\frac{u}{v} = \frac{u^2}{v}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{v}{u^2} & \frac{2u}{v} \\ \frac{1}{u} & -\frac{u^2}{v^2} \end{vmatrix} = \frac{1}{v} - \frac{2}{v} = -\frac{1}{v} \qquad J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|-\frac{1}{v}\right| = \frac{1}{v}$$

Rewrite the integrand: $x^2 y = v$

$$\iint x^2 y\,dA = \int_1^4 \int_1^3 v\,\frac{1}{v}\,du\,dv = (4-1)(3-1) = 6$$

16. (15 points) Compute the surface integral $\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (-x^2 y, xy^2, z^2)$ over the hemisphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$ and upward normal, parametrized as $\vec{R}(\theta,\varphi) = (3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi)$.

$$\begin{matrix} & \\ \vec{e}_\theta = \\ \vec{e}_\varphi = \end{matrix} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3\sin\varphi\sin\theta & 3\sin\varphi\cos\theta & 0 \\ 3\cos\varphi\cos\theta & 3\cos\varphi\sin\theta & -3\sin\varphi \end{vmatrix}$$

$$\begin{aligned} \vec{N} &= \vec{e}_r \times \vec{e}_\theta \\ &= \hat{\imath}(-9\sin^2\varphi\cos\theta) - \hat{\jmath}(9\sin^2\varphi\sin\theta) + \hat{k}(\quad) \\ &\quad + \hat{k}(-9\sin\varphi\cos\varphi\sin^2\theta - 9\sin\varphi\cos\varphi\cos^2\theta) \\ &= (-9\sin^2\varphi\cos\theta, -9\sin^2\varphi\sin\theta, -9\sin\varphi\cos\varphi) \end{aligned}$$

Normal on the hemisphere must point up. So reverse $\vec{N} = (9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi)$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -x^2 y & xy^2 & z^2 \end{vmatrix} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(y^2 - -x^2) = (0,0,x^2+y^2) = (0,0,9\sin^2\varphi)$$

$$\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^{\pi/2}\int_0^{2\pi} \vec{\nabla} \times \vec{F} \cdot \vec{N}\,d\theta\,d\varphi = \int_0^{\pi/2}\int_0^{2\pi} 81\sin^3\varphi\cos\varphi\,d\theta\,d\varphi = 162\pi \frac{\sin^4\varphi}{4}\Bigg|_0^{\pi/2} = \frac{81\pi}{2}$$

17. (25 points) Compute $\iint_S \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (xz, yz, z^2)$ over the complete surface of the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 4$ with outward normal.
(See the figure in #11.)

a. (11 pts) First compute $\iint_P \vec{F} \cdot d\vec{S}$ for the paraboloid $z = x^2 + y^2 \le 4$ which may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & \cos\theta & \sin\theta & 2r \\ \vec{e}_\theta = & -r\sin\theta & r\cos\theta & 0 \end{array} \qquad \begin{aligned} \vec{N} &= \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r) \\ &= (-2r^2\cos\theta, -2r^2\sin\theta, r) \end{aligned}$$

Normal on the paraboloid must point down. So reverse $\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$

$\vec{F} = (xz, yz, z^2) = (r^3\cos\theta, r^3\sin\theta, r^4)$

$\vec{F} \cdot \vec{N} = 2r^5\cos^2\theta + 2r^5\sin^2\theta - r^5 = r^5$

$$\iint_P \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r^5\, dr\, d\theta = 2\pi \frac{r^6}{6}\bigg|_0^2 = \frac{64\pi}{3}$$

b. (11 pts) Second compute $\iint_D \vec{F} \cdot d\vec{S}$ for the disk $x^2 + y^2 \le 4$ with $z = 4$ by writing a parametrization.

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4)$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & \cos\theta & \sin\theta & 0 \\ \vec{e}_\theta = & -r\sin\theta & r\cos\theta & 0 \end{array} \qquad \begin{aligned} \vec{N} &= \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r) \\ &= (0, 0, r) \end{aligned}$$

Normal on the disk must point up. So don't reverse $\vec{N}$.

$\vec{F} = (xz, yz, z^2) = (4r\cos\theta, 4r\sin\theta, 16)$

$\vec{F} \cdot \vec{N} = 16r$

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 16r\, dr\, d\theta = 2\pi 8r^2\big|_0^2 = 64\pi$$

c. (3 pts) Combine the results from (a) and (b) to obtain $\iint_S \vec{F} \cdot d\vec{S}$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_P \vec{F} \cdot d\vec{S} + \iint_D \vec{F} \cdot d\vec{S} = \frac{64\pi}{3} + 64\pi = \frac{256}{3}\pi$$