Name______________________________ Sec__________

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MATH 253H Exam 1 Spring 2009

Sections 200 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. A triangle has vertices at $P = (9,-6,10)$, $Q = (6,6,6)$ and $R = (10,6,3)$. Find the angle at Q.

a. $0°$

b. $30°$

c. $45°$

d. $60°$

e. $90°$ Correct Choice

$\overrightarrow{QP} = P - Q = (-3,12,-4)$ $\quad \overrightarrow{QR} = R - Q = (4,0,-3)$

$\left|\overrightarrow{QP}\right| = \sqrt{9+144+16} = 13$ $\quad \left|\overrightarrow{QR}\right| = \sqrt{16+9} = 5$ $\quad \overrightarrow{QP} \cdot \overrightarrow{QR} = -12+0+12 = 0$

$\cos\theta = 0 \quad \theta = 90°$

2. Find the tangential acceleration of the curve $\vec{r}(t) = (t^2, 2t, \ln t)$.

a. $a_T = 2 + \frac{1}{t^2}$

b. $a_T = 2 - \frac{1}{t^2}$ Correct Choice

c. $a_T = 2t + \frac{1}{t}$

d. $a_T = 2t - \frac{1}{t}$

e. $a_T = t^2 + \ln t$

$\vec{v} = \left(2t, 2, \frac{1}{t}\right) \quad |\vec{v}| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\frac{4t^4+4t^2+1}{t^2}} = \sqrt{\left(\frac{2t^2+1}{t}\right)^2} = \frac{2t^2+1}{t} = 2t + \frac{1}{t}$

$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(2t + \frac{1}{t}\right) = 2 - \frac{1}{t^2}$

3. A jet fighter flies directly East at a constant altitude directly above the Equator.
In what direction does the unit binormal vector $\hat{B}$ point?

a. North Correct Choice
b. South
c. West
d. Up
e. Down

$\hat{T}$ points East. $\quad \hat{N}$ points Down. $\quad$ So $\hat{B} = \hat{T} \times \hat{N}$ points North.

4. Find the plane tangent to the graph of the equation $z\ln(y) + x\ln(z) = 3e^2$ at the point $(x,y,z) = (e^2,e,e^2)$. Write the equation of the plane in the form $z = Ax + By + C$. What is $A \cdot B \cdot C$?

a. $6e^3$

b. $10e^3$

c. $\frac{3}{4}e^3$

d. $\frac{5}{4}e^3$ Correct Choice

e. $4e$

$P = (e^2,e,e^2)$ $\quad \vec{\nabla}F = \left(\ln(z), \frac{z}{y}, \ln(y) + \frac{x}{z}\right)$ $\quad \vec{N} = \vec{\nabla}F\Big|_P = (2,e,2)$ $\quad X = (x,y,z)$

$\vec{N} \cdot X = \vec{N} \cdot P$ $\quad 2x + ey + 2z = 2e^2 + ee + 2e^2 = 5e^2$ $\quad z = -x - \frac{e}{2}y + \frac{5}{2}e^2$

$A \cdot B \cdot C = (-1)\left(-\frac{e}{2}\right)\left(\frac{5}{2}e^2\right) = \frac{5}{4}e^3$

5. Find the plane tangent to the graph of the function $f(x,y) = xy^2 - x^3y$ at the point $(x,y) = (1,2)$. Write the equation of the plane in the form $z = Ax + By + C$. What is $A \cdot B \cdot C$?

a. 3

b. 12 Correct Choice

c. -2

d. -3

e. -12

$f(x,y) = xy^2 - x^3y$ $\quad f(1,2) = 2^2 - 2 = 2$

$f_x(x,y) = y^2 - 3x^2y$ $\quad f_x(1,2) = 2^2 - 3 \cdot 2 = -2$

$f_y(x,y) = 2xy - x^3$ $\quad f_y(1,2) = 2 \cdot 2 - 1 = 3$

$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 2 - 2(x-1) + 3(y-2) = -2x + 3y - 2$

$A \cdot B \cdot C = (-2)(3)(-2) = 12$

6. Consider the function $f(x,y) = 3(x+y)^3$. Find the set of **all** points (x,y) where $\vec{\nabla}f = 0$.

a. The point $(x,y) = (0,0)$.

b. The point $(x,y) = (1,-1)$.

c. The line $(x,y) = (t,t)$.

d. The line $(x,y) = (t,-t)$. Correct Choice

e. The circle $x^2 + y^2 = \frac{1}{9}$.

$\vec{\nabla}f = \left(9(x+y)^2, 9(x+y)^2\right) = 0$ when $x + y = 0$ which is the line $y = -x$
which may be parametrized as $(x,y) = (t,-t)$.

7. The dimensions of a closed rectangular box are measured as 75 cm, 50 cm and 25 cm with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in the calculated surface area of the box.

a. 30 cm^2

b. 60 cm^2

c. 120 cm^2 Correct Choice

d. 150 cm^2

e. 1375 cm^2

$$A = 2xy + 2xz + 2yz$$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz = (2y+2z)dx + (2x+2z)dy + (2x+2y)dz$$

$$= (100+50).2 + (150+50).2 + (150+100).2 = 120$$

8. Find the unit vector direction in which the function $f(x,y) = x^3y^2$ **decreases** most rapidly at the point $(x,y) = (2,-1)$.

a. $\left(-\frac{3}{5}, \frac{4}{5}\right)$ Correct Choice

b. $\left(\frac{3}{5}, -\frac{4}{5}\right)$

c. $\left(-\frac{4}{5}, \frac{3}{5}\right)$

d. $\left(\frac{4}{5}, -\frac{3}{5}\right)$

e. $\left(-\frac{4}{5}, -\frac{3}{5}\right)$

$\vec{\nabla} f = (3x^2y^2, 2x^3y)$ $\quad \vec{v} = -\vec{\nabla} f(2,-1) = -(12,-16) = (-12,16)$ $\quad \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{20}(-12,16) = \left(-\frac{3}{5}, \frac{4}{5}\right)$

9. The point $(0,1)$ is a critical point of the function $f(x,y) = x^2y - 3x^2 + 3y^2 - 2y^3$.
Classify the point $(0,1)$ using the Second Derivative Test.

a. Local Mininum

b. Local Maximum Correct Choice

c. Saddle Point

d. Inflection Point

e. Test Fails

$f_x = 2xy - 6x$ $\quad f_x(0,1) = 0$ $\quad f_y = x^2 + 6y - 6y^2$ $\quad f_y(0,1) = 0$

$f_{xx} = 2y - 6$ $\quad f_{xx}(0,1) = -4 < 0$ $\quad f_{yy} = 6 - 12y$ $\quad f_{yy}(0,1) = -6$ $\quad f_{xy} = 2x$ $\quad f_{xy}(0,1) = 0$

$D = f_{xx}f_{yy} - f_{xy}{}^2$ $\quad D(0,1) = (-4)(-6) - 0^2 = 24 > 0$ Local Maximum

10. Suppose $p = p(x,y)$, while $x = x(u,v)$ and $y = y(u,v)$.

Further, you know the following information:

$$x(1,2) = 3 \qquad y(1,2) = 4 \qquad p(1,2) = 5 \qquad p(3,4) = 6$$

$$\frac{\partial p}{\partial x}(1,2) = 7 \qquad \frac{\partial p}{\partial y}(1,2) = 8 \qquad \frac{\partial p}{\partial x}(3,4) = 9 \qquad \frac{\partial p}{\partial y}(3,4) = 10$$

$$\frac{\partial x}{\partial u}(1,2) = 11 \qquad \frac{\partial x}{\partial v}(1,2) = 12 \qquad \frac{\partial x}{\partial u}(3,4) = 13 \qquad \frac{\partial x}{\partial v}(3,4) = 14$$

$$\frac{\partial y}{\partial u}(1,2) = 15 \qquad \frac{\partial y}{\partial v}(1,2) = 16 \qquad \frac{\partial y}{\partial u}(3,4) = 17 \qquad \frac{\partial y}{\partial v}(3,4) = 18$$

Write out the chain rule for $\frac{\partial p}{\partial v}$. Then use it and the above information to compute $\frac{\partial p}{\partial v}(1,2)$.

a. 134

b. 198

c. 212

d. 249

e. 268 Correct Choice

$$\frac{\partial p}{\partial v} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial p}{\partial x}\bigg|_{(x(u,v),y(u,v))}\frac{\partial x}{\partial v} + \frac{\partial p}{\partial y}\bigg|_{(x(u,v),y(u,v))}\frac{\partial y}{\partial v}$$

$$= \frac{\partial p}{\partial x}(3,4)\frac{\partial x}{\partial v}(1,2) + \frac{\partial p}{\partial y}(3,4)\frac{\partial y}{\partial v}(1,2) = 9 \cdot 12 + 10 \cdot 16 = 268$$

11. A wire has the shape of the helix $\vec{r}(t) = (4\cos t, 4\sin t, 3t)$. It's linear mass density is given by $\rho = x^2 + y^2 + z^2$. Find the total mass of **2 loops** of the wire from $(4,0,0)$ to $(4,0,12\pi)$.

a. $200\pi + 360\pi^3$

b. $320\pi + 960\pi^3$ Correct Choice

c. $320\pi + 48\pi^3$

d. $360\pi + 200\pi^3$

e. $960\pi + 320\pi^3$

$$\vec{v} = (-4\sin t, 4\cos t, 3) \qquad |\vec{v}| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5 \qquad \rho(\vec{r}(t)) = 16\sin^2 t + 16\cos^2 t + 9t^2 = 16 + 9t^2$$

$$M = \int \rho\, ds = \int \rho(\vec{r}(t))|\vec{v}|\, dt = \int_0^{4\pi} (16 + 9t^2)5\, dt = 5\Big[16t + 3t^3\Big]_0^{4\pi}$$

$$= 5(16 \cdot 4\pi + 3 \cdot 64\pi^3) = 320\pi + 960\pi^3$$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (10 points) An object moves around **2 loops** of the helix $\vec{r}(t) = (4\cos t, 4\sin t, 3t)$ from $(4,0,0)$ to $(4,0,12\pi)$ under the action of a force $\vec{F} = (-y, x, z)$.

Find the work done by the force.

$\vec{v} = (-4\sin t, 4\cos t, 3)$ $\quad\quad$ $\vec{F}(\vec{r}(t)) = (-4\sin t, 4\cos t, 3t)$

$$W = \int_0^{4\pi} \vec{F} \cdot d\vec{s} = \int_0^{4\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v}\, dt = \int_0^{4\pi} (16\sin^2 t + 16\cos^2 t + 9t)\, dt = \int_0^{4\pi} (16 + 9t)\, dt$$

$$= \left[16t + \frac{9}{2}t^2 \right]_0^{4\pi} = 64\pi + 72\pi^2$$

13. (10 points) A cylinder is changing in size.

Currently, the radius is $r = 10$ cm and decreasing at 2 cm/sec.

while the height is $h = 25$ cm and increasing at 4 cm/sec.

Find the current volume. Is it increasing or decreasing and at what rate?

The volume is $V = \pi r^2 h = \pi 10^2 25 = 2500\pi$. By the chain rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt} = 2\pi 10 \cdot 25 \cdot (-2) + \pi 10^2 (4) = -600\pi$$ decreasing

14. (10 points) Find the dimensions and volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the paraboloid $9x^2 + 4y^2 + z = 36$.
You must solve by the Lagrange Multiplier method.

Maximize $V = xyz$ subject to the constraint $g = 9x^2 + 4y^2 + z = 36$.

$\vec{\nabla} V = (yz, xz, xy) \qquad \vec{\nabla} g = (18x, 8y, 1)$

Lagrange equations: $yz = \lambda 18x \qquad xz = \lambda 8y \qquad xy = \lambda$

Substitute the third equation into the first two: $yz = 18x^2 y \qquad xz = 8xy^2$

Since $x \neq 0$ and $y \neq 0$, we cancel: $z = 18x^2 \qquad z = 8y^2$

Sustitute into the constraint: $9x^2 = \frac{z}{2} \qquad 4y^2 = \frac{z}{2} \qquad \frac{z}{2} + \frac{z}{2} + z = 36. \qquad 2z = 36$

$z = 18 \qquad 9x^2 = 9 \qquad x = 1 \qquad 4y^2 = 9 \qquad y = \frac{3}{2}$

So the dimensions are $x = 1$, $y = \frac{3}{2}$, $z = 18$, and the volume is $V = 1\left(\frac{3}{2}\right)(18) = 27$.

15. (10 points) Determine if each limit exists. If it exists, find it and prove it. If it does not exist, prove it.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 4y^4}{x^2 + 4y^4}$

Along x-axis: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 4y^4}{x^2 + 4y^4} = \lim_{\substack{y=0 \\ x\to 0}} \frac{x^2 - 4y^4}{x^2 + 4y^4} = \lim_{\substack{y=0 \\ x\to 0}} \frac{x^2 - 0}{x^2 + 0} = 1$

Along y-axis: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 4y^4}{x^2 + 4y^4} = \lim_{\substack{x=0 \\ y\to 0}} \frac{x^2 - 4y^4}{x^2 + 4y^4} = \lim_{\substack{y=0 \\ x\to 0}} \frac{0 - 4y^4}{0 + 4y^4} = -1$

Limit does not exist.

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2}$

Limit does exist. Use polar: $x = r\cos\theta \qquad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{\substack{r\to 0 \\ \theta \text{ arbitrary}}} \frac{r^3 \cos\theta \sin^2\theta}{r^2} = \lim_{\substack{r\to 0 \\ \theta \text{ arbitrary}}} r\cos\theta\sin^2\theta = 0$$

because r goes to zero and $\cos\theta\sin^2\theta$ is bounded between -1 and 1.