MATH 253 EXAM 3 Spring 1998

Sections 501-503 SOLUTIONS P. Yasskin

1. If $F = (xy, yz, xz)$ then $\vec{\nabla} \bullet \vec{F} =$

- **a**. $y - z + x$
- **b**. $(-y, z, -x)$
- **c**. $x + y + z$ correctchoice
- **d**. $(-y, -z, -x)$
- **e**. $-x + y - z$

2. If $F = (xy, yz, xz)$ then $\vec{\nabla} \times \vec{F} =$

- **a**. $y - z + x$
- **b**. $(-y, z, -x)$
- **c**. $x + y + z$
- **d**. $(-y, -z, -x)$ correctchoice
- **e**. $-x + y - z$

3. Compute the line integral $\int y\,dx - x\,dy$ counterclockwise around the semicircle $x^2 + y^2 = 4$ from $(2,0)$ to $(-2,0)$. (HINT: Parametrize the curve.)

- **a**. -4π correctchoice
- **b**. -2π
- **c**. π
- **d**. 2π
- **e**. 4π

4. Compute the line integral $\int \vec{F} \bullet d\vec{s}$ for the vector field $\vec{F} = \left(\frac{1}{x}, \frac{1}{y}\right)$ along the curve $\vec{r}(t) = \left(e^{\cos(t^2)}, e^{\sin(t^2)}\right)$ for $0 \le t \le \sqrt{\pi}$. (HINT: Find a potential.)

- **a**. -2 correctchoice
- **b**. 0
- **c**. $\frac{2}{e}$
- **d**. 1
- **e**. π

5. Compute $\oint (5x + 3y)\,dx + (x - 2y)\,dy$ counterclockwise around the edge of the rectangle $1 \le x \le 5,\ \ 3 \le y \le 6$. (HINT: Use Green's Theorem.)

- **a**. 36
- **b**. 24
- **c**. 12
- **d**. -24 correctchoice
- **e**. -36

6. Compute $\iint\limits_{\partial C} \vec{F} \bullet d\vec{S}$ for the vector field $\vec{F} = \left(zx^3, zy^3, z^2(x^2+y^2)\right)$ over the complete surface of the solid cylinder $C = \left\{(x,y,z) \;|\; x^2+y^2 \le 4,\; 0 \le z \le 3\right\}$.

a. 360π

b. 180π correctchoice

c. 90π

d. 60π

e. 30π

7. If $f(x,y,z) = x\sin(yz) - y\cos(xz) + z\tan(xy)$ then $\vec{\nabla} \times \vec{\nabla} f =$

a. $z\sin(yz) + z\cos(xz) + xy\sec^2(xy)$

b. $\sin(yz) - \cos(xz) + \tan(xy)$

c. $\cos(yz) + \sin(xz) + \sec^2(xy)$

d. 0 correctchoice

e. Does not exist.

8. (25 points) Green's Theorem states that if R is a nice region in the plane and ∂R is its boundary curve traversed counterclockwise then

$$\iint\limits_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint\limits_{\partial R} P\,dx + Q\,dy$$

Verify Green's Theorem if $P = -y^3$ and $Q = x^3$ and R is the region inside the circle $x^2+y^2 = 9$.

8a. (5 points) Compute $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. (HINT: Use rectangular coordinates.)

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial y}(-y^3) = 3x^2 + 3y^2$$

8b. (10 points) Compute $\iint\limits_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$.

a. (HINT: Switch to polar coordinates and don't forget the Jacobian.)

$$\iint\limits_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint\limits_{R} 3(x^2+y^2)\; dx\,dy = 3\int_0^{2\pi}\int_0^3 (r^2)\; r\; dr\,d\theta$$

$$= 3(2\pi)\left[\frac{r^4}{4}\right]_{r=0}^{3} = \frac{243\pi}{2}$$

8c. (10 points) Compute $\oint\limits_{\partial R} P\,dx + Q\,dy$. (HINT: Parametrize of the boundary circle.)

$\vec{r}(t) = (3\cos t, 3\sin t) \qquad \vec{v}(t) = (-3\sin t, 3\cos t)$

$\vec{F} = (P,Q) = (-y^3, x^3) = \left(-27\sin^3 t, 27\cos^3 t\right) \qquad \vec{F} \bullet \vec{v} = 81\sin^4 t + 81\cos^4 t$

$$\oint\limits_{\partial R} P\,dx + Q\,dy = 81\int_0^{2\pi}\left(\sin^4 t + \cos^4 t\right) dt = 81\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)^2 + \left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \frac{81}{4}\int_0^{2\pi}(1 - 2\cos 2t + \cos^2 2t) + (1 + 2\cos 2t + \cos^2 2t)\,dt$$
$$= \frac{81}{4}\int_0^{2\pi}(2 + 2\cos^2 2t)\,dt = \frac{81}{4}\int_0^{2\pi}(2 + 1 + \cos 4t)\,dt = \frac{81}{4} \bullet 3 \bullet 2\pi = \frac{243\pi}{2}$$

9. (30 points) Stokes' Theorem states that if S is a surface in 3-space and ∂S is its boundary curve traversed counterclockwise as seen from the tip of the normal to S then

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \oint_{\partial S} \vec{F} \bullet d\vec{s}$$

Verify Stokes' Theorem if $F = (-yx^2, xy^2, x^2 + y^2)$ and S is the part of the cone $z = \sqrt{x^2 + y^2}$ below $z = 2$ with normal pointing in and up.

9a. (5 points) Compute $\vec{\nabla} \times \vec{F}$. (HINT: Use rectangular coordinates.)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -yx^2 & xy^2 & x^2 + y^2 \end{vmatrix} = i(2y - 0) - j(2x - 0) + k(y^2 - -x^2) = (2y, 2x, x^2 + y^2)$$

9b. (10 points) Compute $\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S}$.

a. (HINT: Here is the parametrization of the cone and the steps you should use. Remember to check the orientation of the surface.)

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

$\vec{R}_r = (\cos\theta, \sin\theta, 1)$

$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = i(-r\cos\theta) - j(r\sin\theta) + k(r) = (-r\cos\theta, -r\sin\theta, r)$

$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(r,\theta)\right) = (2r\sin\theta, -2r\cos\theta, r^2)$

$\left(\vec{\nabla} \times \vec{F}\right) \bullet \vec{N} = -2r^2\sin\theta\cos\theta - 2r^2\cos\theta\sin\theta + r^3 = r^3$

$$\iint_S \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = 2\pi\left[\frac{r^4}{4}\right]_0^2 = 8\pi$$

9c. (15 points) Compute $\oint_{\partial S} \vec{F} \bullet d\vec{s}$. Recall $F = (-yx^2, xy^2, x^2 + y^2)$.

(HINT: Parametrize the boundary circle. Remember to check the orientation of the curve.)

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2)$ $\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$

$$\vec{F}\left(\vec{r}(\theta)\right) = \left(-2\sin\theta \bullet 4\cos^2\theta, 2\cos\theta \bullet 4\sin^2\theta, 4\cos^2\theta + 4\sin^2\theta\right) = \left(-8\sin\theta\cos^2\theta, 8\cos\theta\sin^2\theta,\right.$$

$$\vec{F} \bullet \vec{v} = 16\sin^2\theta\cos^2\theta + 16\cos^2\theta\sin^2\theta = 32\sin^2\theta\cos^2\theta$$

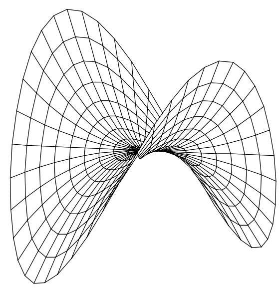
$$\oint_{\partial S} \vec{F} \bullet d\vec{s} = \int_0^{2\pi} 32\sin^2\theta\cos^2\theta \, d\theta = \int_0^{2\pi} 8\sin^2(2\theta)\, d\theta = 8 \bullet \frac{1}{2}(2\pi - 0) = 8\pi$$

10. (10 points)

The spider web at the right is the graph of the hyperbolic paraboloid $z = xy$.
It may be parametrized as

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2\sin\theta\cos\theta).$$

Find the area of the web for $r \le \sqrt{8}$.



$$\vec{R}_r = (\cos\theta, \quad \sin\theta, \quad 2r\sin\theta\cos\theta)$$

$$\vec{R}_\theta = \left(-r\sin\theta, r\cos\theta, r^2\left(\cos^2\theta - \sin^2\theta\right)\right)$$

$$\begin{aligned}\vec{N} &= i\left(r^2\sin\theta\left(\cos^2\theta - \sin^2\theta\right) - 2r^2\sin\theta\cos^2\theta\right) - j\left(r^2\cos\theta\left(\cos^2\theta - \sin^2\theta\right) - -2r^2\sin^2\theta\cos\theta\right) \\ &\quad + k\left(r\cos^2\theta - -r\sin^2\theta\right) = \left(-r^2\sin\theta\cos^2\theta - r^2\sin^3\theta, -r^2\cos^3\theta - r^2\sin^2\theta\cos\theta, r\right) \\ &= (-r^2\sin\theta, -r^2\cos\theta, r)\end{aligned}$$

$$\left|\vec{N}\right| = \sqrt{r^4\sin^2\theta + r^4\cos^2\theta + r^2}\ \sqrt{r^4 + r^2} = r\sqrt{r^2+1}$$

$$\begin{aligned}A &= \int_0^{2\pi}\int_0^{\sqrt{8}} \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{8}} r(r^2+1)^{1/2}\, dr\, d\theta = 2\pi\left[\frac{2}{3}\frac{(r^2+1)^{3/2}}{2}\right]_0^{\sqrt{8}} \\ &= \frac{2\pi}{3}\left[(9)^{3/2} - (1)^{3/2}\right] = \frac{2\pi}{3}[27-1] = \frac{52\pi}{3}\end{aligned}$$